1. Let $(x_0,y_0)\in E^2$ s.t. $x_0^2+y_0^2<4$.

IF $(x,y)\in N_\delta(x_0,y_0)$ then $|(x,y)| \le |(x-x_0,\, y-y_0)| + |(x_0,y_0)|$ (triangle ineq.)

$$< \delta + \underbrace{\sqrt{x_0^2+y_0^2}}_{<2}$$

Pick $\delta = \dfrac{2-\sqrt{x_0^2+y_0^2}}{2} > 0$, so $|(x,y)| < 2$,

i.e. $(x,y)\in S$.

We proved that $N_\delta(x_0,y_0)\subset S$ so S is open □

2. a) $\lim\limits_{x\to 0} f(x,0) = \lim\limits_{x\to 0} \dfrac{x^4}{x^4} = 1 \ne \lim\limits_{y\to 0} f(0,y) = 0$ so

$\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist.

b) $0\le |f(x,y)-0| \le \left|\dfrac{x^4}{x^2}\right| \le x^2 \xrightarrow[(x,y)\to 0]{} 0$ so $f(x,y) \xrightarrow[(x,y)\to(0,0)]{} 0$ by the squeeze theorem.

c) $f(x,y) = \dfrac{\sin(x^2+y^2)}{\cos(x^4+y^4)}$ $\quad \sin(x^2+y^2) \xrightarrow{(x,y)\to(0,0)} \sin(0)=0$ by continuity of sin

$\cos(x^4+y^4) \xrightarrow[(x,y)\to(0,0)]{} \cos(0)=1 \ne 0$ ——— cos

so by the quotient rule, $f(x,y) \xrightarrow[(x,y)\to(0,0)]{} \dfrac{0}{1} = 0$.

d) $f(x,y) = \dfrac{xy(x+y)(y-x)(y-2x)(y+2x)}{x^4+y^4}$

Using $|x|, |y| \le (x^4+y^4)^{1/4}$ one gets

$$0\le |f(x,y)| \le \frac{(x^4+y^4)^{2/4}\, 2(x^4+y^4)^{1/4} (y-x)(y-2x)\, 3(x^4+y^4)^{1/4}}{x^4+y^4}$$

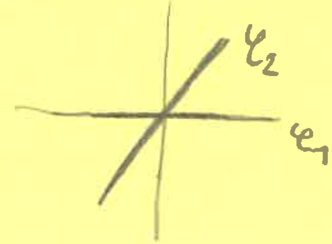
$$= 6(y-x)(y-2x)$$

$$\xrightarrow[(x,y)\to(0,0)]{} 0$$

By the squeeze theorem, $f(x,y) \xrightarrow[(x,y)\to(0,0)]{} 0$

e) $\lim_{x\to 0} f(x,0) = 0 \neq \lim_{x\to 0} f(x,3x) = \frac{3(9-1)(9-4)}{1+36}$

so does not exist.



3) a) $|f(x,y)| = \frac{7x^6}{x^4+y^4}$

$\leq 7x^2 \frac{x^4}{x^4+y^4} \leq 7x^2 \leq 7(x^2+y^2) < \varepsilon$

provided $x^2+y^2 < \frac{\varepsilon}{7}$, i.e. provided $\sqrt{x^2+y^2} < \delta = \sqrt{\frac{\varepsilon}{7}}$ □

b) $|f(x,y)| \leq \frac{7|x|^3 y^2}{x^4+y^4} \leq 7\frac{(x^4+y^4)^{3/4}(x^4+y^4)^{2/4}}{x^4+y^4} = 7(x^4+y^4)^{1/4}$

$\leq 7(x^4+2x^2y^2+y^4)^{1/4}$

$= 7(x^2+y^2)^{1/2}$

$< \varepsilon$

provided $\sqrt{x^2+y^2} < \delta = \frac{\varepsilon}{7}$.

4) As products, sums and compositions of such functions, F has continuous partial derivatives (up to any order) on the whole $\mathbb{E}^3$

$$JF(x,y,z) = \begin{pmatrix} 2xy^2e^{x^2} & 2ye^{x^2} & -1 \\ y\sin(z) & x\sin(z) & xy\cos(z) \\ 0 & e^y & -\sin(z) \end{pmatrix}.$$

5) a) $\frac{f(\Delta x,0)-f(0,0)}{\Delta x} = \frac{0-0}{\Delta x} = 0 \xrightarrow[\Delta x\to 0]{} 0$ so $\frac{\partial f}{\partial x}(0,0) = 0$

$\frac{f(0,\Delta y)-f(0,0)}{\Delta y} = 0 \xrightarrow[\Delta y\to 0]{} 0$ so $\frac{\partial f}{\partial y}(0,0) = 0$

b) $\frac{f(\Delta x,0)-f(0,0)}{\Delta x} = \frac{\frac{(\Delta x)^4}{(\Delta x)^4} - 0}{\Delta x} = \frac{1}{\Delta x} \xrightarrow[\Delta x\to 0^\pm]{} \pm\infty$ so $\frac{\partial f}{\partial x}(0,0)$ does not exist.

$\frac{f(0,\Delta y)-f(0,0)}{\Delta y} = \frac{0-0}{\Delta y} = 0 \xrightarrow[\Delta y\to 0]{} 0$ so $\frac{\partial f}{\partial y}(0,0) = 0$

c) $\dfrac{f(\Delta x,0)-f(0,0)}{\Delta x} = 0 \xrightarrow[\Delta x\to 0]{} 0$ so $\dfrac{\partial f}{\partial x}(0,0)=0$.

$$\frac{f(0,\Delta y)-f(0,0)}{\Delta y} = \frac{\frac{|\Delta y|^5}{(\Delta y)^4}-0}{\Delta y} = \frac{|\Delta y|^5}{(\Delta y)^5} = \begin{cases}+1 & \text{if } \Delta y>0\\ -1 & \text{if } \Delta y<0\end{cases}$$

has no limit as $\Delta y \to 0$, so $\dfrac{\partial f}{\partial y}(0,0)$ does not exist.

Reminder: For limits of the type $\dfrac{P(x,y)}{Q(x,y)}$ where P, Q are polynomials with $P(0,0)=Q(0,0)=0$, a good rule of thumb is to look at the total degree of the dominant term of P vs the dominant term of Q ($>$ indicates hope for a 0-limit, ≤ 0 for non existence).

6) f has cont. partial derivatives up to order 1 so is differentiable, and

$$f(x+\Delta x, y+\Delta y) = f(x,y) + \overset{\frac{\partial f}{\partial x}}{y^3}\,\Delta x + \overset{\frac{\partial f}{\partial y}}{3xy^2}\,\Delta y + \varepsilon_1(x,y,\Delta x,\Delta y)\,\Delta x + \varepsilon_2(x,y,\Delta x,\Delta y)\,\Delta y$$

So

$$(x+\Delta x)(y+\Delta y)^3 = \cancel{xy^3} + \cancel{y^3\Delta x} + \cancel{3xy^2\Delta y} + \varepsilon_1\Delta x + \varepsilon_2\Delta y$$

$$\|$$

$$(x+\Delta x)(y^3+3y^2\Delta y+3y\Delta y^2+\Delta y^3)$$

$$\|$$

$$\cancel{xy^3} + \cancel{3xy^2\Delta y} + \underbrace{3xy\Delta y^2 + \Delta y^3 x}_{\varepsilon_2\Delta y} + \cancel{\Delta x\, y^3} + \underbrace{3y^2\Delta x\Delta y + 3y\Delta y^2\Delta x + \Delta x\Delta y^3}_{\varepsilon_1\Delta x}$$

So that

$$\left|\begin{aligned} \varepsilon_1 &= 3y^2\Delta y + 3y\Delta y^2 + \Delta y^3 \\ \varepsilon_2 &= 3xy\Delta y + x\Delta y^2 \end{aligned}\right.$$

fit and indeed,

$$\varepsilon_1, \varepsilon_2 \xrightarrow[(\Delta x,\Delta y)\to(0,0)]{} 0.$$

$\longrightarrow$

7) $$\frac{\partial z}{\partial t}(s,t) = \frac{\partial f}{\partial x}(1-s^2-t^2, t^3+s^3)\frac{\partial x}{\partial t} + \frac{\partial F}{\partial y}(1-s^2-t^2, t^3+s^3)\frac{\partial y}{\partial t}$$

$$= -2t\,\frac{\partial F}{\partial x}(1-s^2-t^2, t^3+s^3) + 3t^2\,\frac{\partial F}{\partial y}(1-s^2-t^2, t^3+s^3)$$

At $s=0, t=1$ one gets $\frac{\partial z}{\partial t}(0,1) = -2\frac{\partial F}{\partial x}(0,1) + 3\frac{\partial F}{\partial y}(0,1)$

$$= -2\times 8 + 3\times 9 = 11.$$

When $s=t=0$, $\frac{\partial z}{\partial t}(0,0) = 0$.

8) a) Assuming $w = w(y,z)$ & $x = x(y,z)$; taking $\frac{\partial}{\partial y}$ we get

$$\begin{cases} 0 = \frac{\partial x}{\partial y}y + x + 2z^2 w\frac{\partial w}{\partial y} & (1) \\ 0 = \frac{\partial x}{\partial y}z + 2yw\frac{\partial w}{\partial y} + 2yw^2 & (2) \end{cases}$$

Taking $z*(1) - y*(2)$ one gets

$$0 = xz + 2z^3 w\frac{\partial w}{\partial y} - 2y^2 w\frac{\partial w}{\partial y} - 2y^2 w^2$$

so $$\boxed{\frac{\partial w}{\partial y}(y,z) = \frac{2w^2y^2 - xz}{2w(z^3-y^2)}}$$

b) observe that at $(1,0,1,1)$ the denom. up there is 0.

Indeed, our assumption cannot be done there, because if we call

$$F(w,x,y,z) = xy + z^2w^2 - 1$$
$$G(\quad) = x^2 + y^2w^2 - 1$$

then one has there

$$\begin{pmatrix} \frac{\partial F}{\partial w} & \frac{\partial F}{\partial x} \\ \frac{\partial G}{\partial w} & \frac{\partial G}{\partial x} \end{pmatrix} = \begin{pmatrix} 2wz^2 & y \\ 2wy^2 & z \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix}$$

has <u>determinant 0</u>. The IFT cannot apply in these variables there.

→

9) · $\boxed{\text{On } x>0}$, $f(x,y) = \dfrac{x^5}{x^4+y^2}$ and $\dfrac{\partial f}{\partial x}$, $\dfrac{\partial f}{\partial y}$ exist by diff. of a quotient whose denom. does not cancel.

· $\boxed{\text{Same on } x<0}$ where $f(x,y) = \dfrac{-x^5}{x^4+y^2}$.

· $\boxed{\text{At a point } (0,y) \text{ with } y\neq 0}$

$$\frac{f(\Delta x,y)-f(0,y)}{\Delta x} = \frac{\frac{|\Delta x|^5}{(\Delta x)^4+y^2}-0}{\Delta x} = \frac{\pm(\Delta x)^4}{(\Delta x)^4+y^2} \xrightarrow[\Delta x\to 0]{} 0 \text{ since } y\neq 0$$

so there $\dfrac{\partial f}{\partial x}(0,y) = 0$.

$$\frac{f(0,y+\Delta y)-f(0,y)}{\Delta y} = \frac{0-0}{\Delta y} = 0 \xrightarrow[\Delta y\to 0]{} 0 \quad \text{so} \quad \frac{\partial f}{\partial y}(0,y) = 0$$

· $\boxed{\text{At } (0,0)}$: $\dfrac{f(\Delta x,0)-f(0,0)}{\Delta x} = \dfrac{\frac{|\Delta x|^5}{\Delta x^4}-0}{\Delta x} = \operatorname{sign}(\Delta x)$

does not have a limit as $\Delta x\to 0$ so $\dfrac{\partial f}{\partial x}(0,0)$ does not exist.

$$\frac{f(0,\Delta y)-f(0,0)}{\Delta y} = 0 \xrightarrow[\Delta y\to 0]{} 0 \quad \text{so} \quad \frac{\partial f}{\partial y}(0,0) = 0$$

10) F and G are both $\mathcal{C}^1$. Moreover

$$\begin{pmatrix} F_w & F_x & F_y & F_z \\ G_w & G_x & G_y & G_z \end{pmatrix}_{|(0,0,0,0)}$$

$$= \begin{pmatrix} 1+xyz\cos(wxyz) & 1+wyz\cos(wxyz) & 1+wxz\cos(wxyz) & 1+wxy\cos(wxyz) \\ 1-xyz\sin(wxyz) & 1-wyz\sin(wxyz) & 2-wxz\sin(wxyz) & 2-wxy\sin(wxyz) \end{pmatrix}_{|(0,0,0,0)}$$

$$= \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 2 \end{pmatrix},$$

· Solving for w & z, $\det\begin{pmatrix} F_w & F_y \\ G_w & G_y \end{pmatrix}_{|(0,0)} = \det\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \neq 0$, thm applies

· For x & y, $\det\begin{pmatrix} F_x & F_y \\ G_x & G_y \end{pmatrix}_{|(0,0)} = \det\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \neq 0$, yes.

For x&z, $\det\begin{pmatrix} F_x & F_z \\ G_x & G_z \end{pmatrix}_{|(0,0)} = \det\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \neq 0$, yes.

For y&z, $\det\begin{pmatrix} F_y & F_z \\ G_y & G_z \end{pmatrix}_{|(0,0)} = \det\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} = 0$, thm does not apply.

11) f is $\mathcal{C}^1$ on $\mathbb{E}^2$ and $\vec{\nabla} f(x,y) = \begin{pmatrix} 6xy - 12x \\ 3y^2 + 3x^2 - 12y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

if and only if $\begin{cases} 6x(y-2) = 0 \\ 3y(y-4) = -3x^2 \end{cases}$

Possible solutions: $x = 0$, $y = 0$ or 4.
$y = 2$, $-3x^2 = -12$ so $x = \pm 2$.

$\to$ $(0,0)$; $(0,4)$; $(2,2)$; $(-2,2)$

$Hf(x,y) = \begin{pmatrix} 6y-12 & 6x \\ 6x & 6y-12 \end{pmatrix}$

At $(0,0)$; $Hf(0,0) = \begin{pmatrix} -12 & 0 \\ 0 & -12 \end{pmatrix}$, eigenvalues are $-12 < 0 \rightsquigarrow$ $(0,0)$ is a pt of rel. max.

At $(0,4)$; $Hf(0,4) = \begin{pmatrix} 12 & 0 \\ 0 & 12 \end{pmatrix}$ —— > 0 $(0,4)$ —— min

At $(2,2)$ $Hf(2,2) = \begin{pmatrix} 0 & 12 \\ 12 & 0 \end{pmatrix} = M$

$\det(M - \lambda I_2) = \begin{vmatrix} -\lambda & 12 \\ 12 & -\lambda \end{vmatrix} = \lambda^2 - 12^2$, eigenvalues are ± 12, so $(2,2)$ is a saddle pt.

At $(-2,2)$, $Hf(-2,2) = \begin{pmatrix} 0 & -12 \\ -12 & 0 \end{pmatrix}$

$\det(\quad - \lambda I_2) = \begin{vmatrix} -\lambda & -12 \\ -12 & -\lambda \end{vmatrix} = \lambda^2 - 12^2$, same as