21-268 (Spring) Multidimensional Calculus Lec.1
Midterm #2

Laurent Dietrich   Carnegie Mellon University

April 5, 2017

This exam has 6 independent questions. You are only required to chose and answer 5 of them to get a full 50/50. If you do more, one third of your lowest question score will be added as bonus to your grade.

No textbook, notes, electronic device, recitation or exterior material is authorized. Please use a pen and write your Andrew ID and page numbers on any additional sheet you should join.

All answers should be justified, unless otherwise stated. Clarity of your statements and care taken in your writing and presentation will be taken into account when grading.

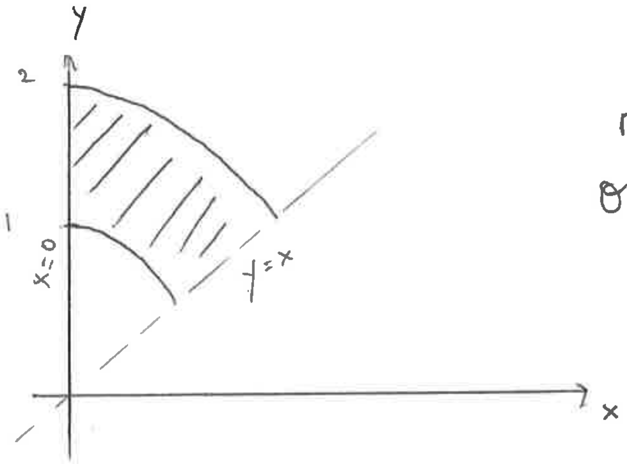
Time: 50 minutes

Name	
Andrew ID	
Section	

Question	1	2	3	4	5	6	Total
Points	10	10	10	10	10	10	50
Score							

1. (10 points) Let R be the region in the first quadrant of the plane that is bounded by the curves $y = x$, $x = 0$, $x^2 + y^2 = 1$, and $x^2 + y^2 = 4$. Compute the integral

$$\int\int_R (x^2 + y^2)^{1/4} x \, dA.$$



$r \in (1,2)$

$\theta \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$

$$\iint_R (x^2+y^2)^{1/4} x \, dA$$

$$= \int_1^2 \int_{\pi/4}^{\pi/2} r^{1/2} \, r\cos\theta \; r \, d\theta \, dr$$

$$= [\sin\theta]_{\pi/4}^{\pi/2} \left[\frac{2}{7} r^{7/2}\right]_1^2$$

$$= \frac{2}{7}\left(1 - \frac{\sqrt{2}}{2}\right)\left(2^{7/2} - 1\right).$$

2. Let $f(x) = \frac{1}{x^2}$ and $[a,b] = [0.1, 2]$. Assume that $x_0 = 0.1 < x_1 < \cdots < x_n = 2$ and let $h = \max(x_1 - x_0, x_2 - x_1, \ldots, x_n - x_{n-1})$.

(a) (4 points) Let $m_i = \sqrt{x_{i-1}x_i}$. Show that

$$\sum_{i=1}^{n} f(m_i)(x_i - x_{i-1}) = 9.5.$$

$$\sum_{i=1}^{n} \frac{x_i - x_{i-1}}{x_{i-1}x_i} = \sum_{i=1}^{n} \frac{1}{x_{i-1}} - \frac{1}{x_i} = \frac{1}{x_0} - \cancel{\frac{1}{x_1}} + \cancel{\frac{1}{x_1}} - \cancel{\frac{1}{x_2}} + \cancel{\ldots} - \frac{1}{x_n} = \frac{1}{x_0} - \frac{1}{x_n}$$

$$= \frac{1}{a} - \frac{1}{b}$$

$$= 10 - 1/2$$

$$= 9.5$$

(b) (6 points) Let $x_i^* \in [x_{i-1}, x_i]$ for $i = 1, \ldots, n$. For any $\epsilon > 0$ find $\delta > 0$ such that $0 < h < \delta$ implies

$$\left| \sum_{i=1}^{n} f(x_i^*)(x_i - x_{i-1}) - 9.5 \right| < \epsilon.$$

Remark for fun: this proves that $\int_{0.1}^{2} x^{-2} \mathrm{d}x = 9.5 = [-1/x]_{0.1}^{2}$ *as we already knew.*

$$\left| \sum_{i=1}^{n} f(x_i^*)(x_i - x_{i-1}) - 9.5 \right|$$

$$= \left| \sum_{i=1}^{n} \frac{1}{x_i^{*2}}(x_i - x_{i-1}) - \sum_{i=1}^{n} \frac{1}{x_{i-1}x_i}(x_i - x_{i-1}) \right|$$

$$= \left| \sum_{i=1}^{n} \underbrace{\left(\frac{1}{x_i^{*2}} - \frac{1}{x_{i-1}x_i}\right)}_{|\;\;\;|} (x_i - x_{i-1}) \right| \quad (*)$$

$$|\;\;\;| \leq \frac{x_{i-1}x_i - (x_i^*)^2}{(x_i^{*2})(x_{i-1}x_i)} \leq \frac{x_{i-1}x_i - (x_i^*)^2}{a^4}$$

$$\leq \frac{x_i^2 - (x_{i-1})^2}{a^4}$$

$$\leq \frac{(x_i - x_{i-1})(x_i + x_{i-1})}{a^4} \quad \leq 4$$

$$\leq \frac{4h}{a^4}$$

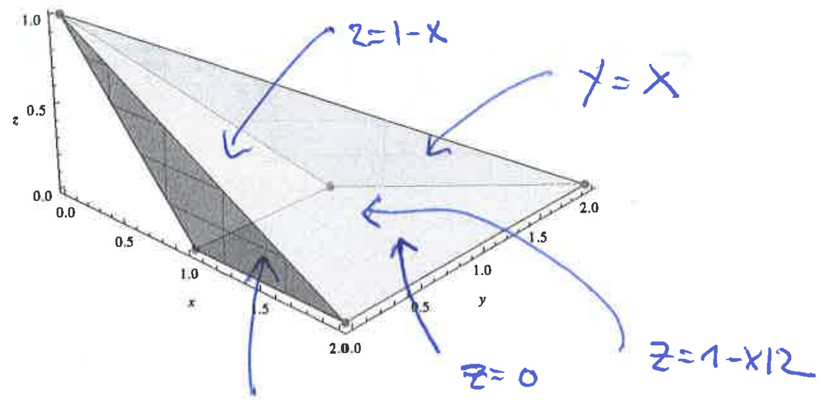
So $(*) \leq \frac{4h}{a^4} \sum_{i=1}^{n} x_i - x_{i-1} = \frac{4h}{a^4}(b-a)$.

$$\boxed{\text{So } (*) < \varepsilon \text{ provided } h < \frac{a^4 \varepsilon}{4(b-a)}}$$

3. Let R be the region in the first octant (i.e. $x, y, z \geq 0$) bounded by the planes

$$z = 1 - x, \quad z = 1 - x/2, \quad y = 0, \quad y = x.$$

(a) (2 points) Below is a picture or R. Identify **clearly** to which plane each face corresponds



(b) (8 points) Compute $\int\int\int_R x dV$.

The easiest order seems to be $\iiint \cdots \, dy\, dx\, dz$ (or $dx\,dy\,dz$)

- z takes values between 0 and 1.
- Once z is fixed there, x goes between 1 and $2-2z$
- Once x is fixed there, y ——— 0 and x.

So $\iiint_R x\, dV = \int_0^1 \int_{1-z}^{2-2z} \int_0^x x \, dy\, dx\, dz$

$$= \int_0^1 \int_{1-z}^{2-2z} x^2 \, dx\, dz = \frac{1}{3}\int_0^1 \left((2-2z)^3 - (1-z)^3\right) dz$$

$$= \frac{7}{3}\int_0^1 (1-z)^3\, dz$$

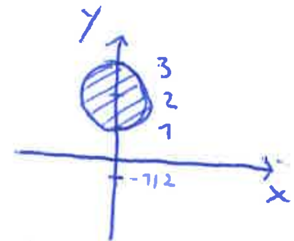
$$= -\frac{7}{12}\left[(1-z)^4\right]_0^1 = \frac{7}{12}.$$

4. Let $f(x,y) = x^2 + 2y^2 + 2y$ and $D = \{(x,y) : x^2 + (y-2)^2 \leq 1\}$.

(a) (2 points) Why does f have points of absolute minimum and absolute maximum on D?

Because f is continuous on D compact (closed & bdd).

(b) (8 points) Find them and their corresponding values.



- We first look for critical points. $\vec{\nabla} f(x,y) = \begin{pmatrix} 2x \\ 4y+2 \end{pmatrix}$

so critical points are $\{(0, -\frac{1}{2})\}$ but this is not in D so extrema cannot be reached in the interior of D.

- We now investigate the boundary of D: $x^2 + (y-2)^2 = 1$, can be parametrized by $x = \pm\sqrt{1-(y-2)^2}$, $y \in [1,3]$.

There, $f(x,y) = f(\pm\sqrt{1-(y-2)^2}, y) = 1-(y-2)^2 + 2y^2 + 2y$

$= 1 - y^2 + 4y - 4 + 2y^2 + 2y$

$= -3 + 6y + y^2.$

$=: g(y)$

$g'(y) = 6 + 2y$

y	1	3
$g'(y)$	+	
$g(y)$	$\nearrow$	

So it only remains

$\min_D f = g(1) = f(0,1)$

$= 4.$

$\max_D f = g(3) = f(0,3)$

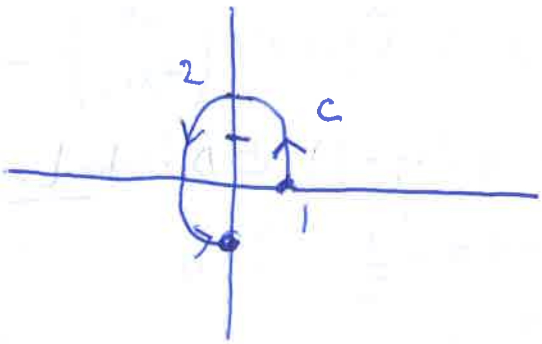
$= 24.$

5. (10 points) Let

$$C = \left\{ (x,y) : x^2 + \frac{y^2}{4} = 1 \right\} \setminus \{(x,y) : x > 0 > y\}.$$

oriented from $(1,0)$ towards $(0,-2)$. Let $\vec{F}(x,y) = xy\vec{i} + 3y^2\vec{j}$. Compute

$$\int_C \vec{F} \cdot d\vec{r}.$$



Param. of C: $\vec{r}(\theta) = \overbrace{\cos\theta}^{x(\theta)}\vec{i} + \overbrace{2\sin\theta}^{y(\theta)}\vec{j}$, $\theta \in (0, \frac{3\pi}{2})$

so that $\int_C \vec{F} \cdot d\vec{r} = \int_C xy\,dx + \int_C 3y^2\,dy$

$$= \int_0^{3\pi/2} 2\cos\theta\sin\theta \underbrace{(-\sin\theta)}_{x'(\theta)} d\theta + \int_0^{3\pi/2} 12\sin^2\theta \times \underbrace{2\cos\theta}_{y'(\theta)} d\theta$$

$$= 2\int_0^{3\pi/2} \cos\theta\sin\theta(-\sin\theta + 12\sin\theta)\,d\theta$$

$$= 22\int_0^{3\pi/2} \underbrace{\cos\theta}_{u'} \underbrace{\sin^2\theta}_{u^2}\,d\theta = \frac{22}{3}\left[\sin^3\theta\right]_0^{3\pi/2} = -\frac{22}{3}.$$

6. (10 points) Assume that $g(t,x)$ defines a C^1 function and let

$$u(t,x) = \int_{xe^{-t}}^{x} \underbrace{g\left(t + \ln(y) - \ln(x), y\right)\frac{1}{y}}_{\text{called } f(t,y,x)}\mathrm{d}y$$

for $t \in \mathbb{R}$, $x > 0$. Show that

$$\frac{\partial u}{\partial t}(t,x) + x\frac{\partial u}{\partial x}(t,x) = g(t,x).$$

Remark for fun: in the context of partial differential equations, we solved a 1D transport equation with velocity x, with initial data $u(0,\cdot) \equiv 0$ and source term g.

By the short extension of Leibniz's rule as in HW#10, applied twice

x is fixed here

$$\frac{\partial u}{\partial t}(t,x) = \underbrace{\frac{\partial x}{\partial t}}_{=0} \times (\quad) - \frac{\partial (xe^{-t})}{\partial t} \underbrace{g\left(t+\ln(xe^{-t})-\ln(x), xe^{-t}\right)\frac{1}{xe^{-t}}}_{\text{what is inside the integral with } y = xe^{-t}\text{, ie } f(t, xe^{-t}, x)}$$

$$+ \int_{xe^{-t}}^{x} \frac{\partial}{\partial t}\left(g(t+\ln(y)-\ln(x), y)\frac{1}{y}\right) dy$$

$$= \cancel{xe^{-t}}\, g(t-t, xe^{-t})\frac{1}{\cancel{xe^{-t}}} + \int_{xe^{-t}}^{x} \frac{\partial g}{\partial t}\left(t+\ln\left(\tfrac{y}{x}\right), y\right)\frac{1}{y}\, dy.$$

t is fixed here

$$\frac{\partial u}{\partial x}(t,x) = 1 \times \underbrace{g(t,x)\frac{1}{x}}_{f(t,x,x)} - \cancel{e^{-t}}\, g(t-t, xe^{-t})\frac{1}{x\cancel{e^{-t}}} \quad f(t, xe^{-t}, x)$$

$$+ \int_{xe^{-t}}^{x} \frac{\partial}{\partial x}\left(g(t+\ln(y)-\ln(x)), y\right)\frac{1}{y}\, dy$$

$$= \frac{g(t,x)}{x} - \frac{g(0, xe^{-t})}{x} + \int_{xe^{-t}}^{x} \frac{\partial g}{\partial t}\left(t+\ln\left(\tfrac{y}{x}\right), y\right) \times \left(-\frac{1}{x}\right)\frac{1}{y}\, dy$$

So $\left(\frac{\partial u}{\partial t} + x\frac{\partial u}{\partial x}\right)(t,x) = \cancel{g(0,xe^{-t}) - g(0,xe^{-t})} + \cancel{\int - \int}$

$+ g(t,x) \quad = g(t,x)$