[illegible]
Section [illegible] - 21-268
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1) $f(x,y,z) = \begin{pmatrix} xy^2z^3e^{\sqrt{xy}} \\ xy\sin(yz) \end{pmatrix} = \begin{pmatrix} f_1(x,y,z) \\ f_2(x,y,z) \end{pmatrix}$

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$\frac{\partial f_1}{\partial x} = y^2z^3e^{\sqrt{xy}} + xy^2z^3e^{\sqrt{xy}} \cdot \frac{\sqrt{y}}{2\sqrt{x}}$ ✓ x, y are same sign, $x \neq 0$ ✓

$\frac{\partial f_1}{\partial y} = 2xyz^3e^{\sqrt{xy}} + xy^2z^3e^{\sqrt{xy}} \cdot \frac{\sqrt{x}}{2\sqrt{y}}$ x, y same sign, $y \neq 0$

$\frac{\partial f_1}{\partial z} = 3xy^2z^2e^{\sqrt{xy}}$ ✓ x, y same sign

$\frac{\partial f_2}{\partial x} = y\sin(yz)$ ✓ $(x,y,z) \in (\mathbb{R}^3)$

$\frac{\partial f_2}{\partial y} = x\sin(yz) + xy\cos(yz) \cdot z$ ✓

$\frac{\partial f_2}{\partial z} = xy\cos(yz) \cdot y$ ✓

The domain is x, y are the same sign and $x, y \neq 0$. $z \in \mathbb{R}$

$$J = \begin{pmatrix} y^2z^3e^{\sqrt{xy}} + xy^2z^3e^{\sqrt{xy}} \cdot \frac{\sqrt{y}}{2\sqrt{x}}, & 2xyz^3e^{\sqrt{xy}} + xy^2z^3e^{\sqrt{xy}} \cdot \frac{\sqrt{x}}{2\sqrt{y}}, & 3xy^2z^2e^{\sqrt{xy}} \\ y\sin(yz) & x\sin(yz) + xyz\cos(yz) & xy^2\cos(yz) \end{pmatrix}$$

2) $F(x,y) = \begin{pmatrix} e^x\cos(y) \\ e^x\sin(y) \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$

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a) (x,y)

$\vec{F}: D \to \mathbb{E}^2$ and $D \in \mathbb{E}^2$

The function is defined in the Domain D in $\mathbb{E}^n$.
f_1, f_2 are both continuous and have partial derivatives that are continuous, at all (x,y).
The Jacobian is

$$J = \begin{pmatrix} e^x\cos(y) & -e^x\sin(y) \\ e^x\sin(y) & e^x\cos(y) \end{pmatrix}$$

$e^{2x}(\cos^2(y)+\sin^2 y)$

$\det(J) = e^{2x}\cos^2(y)+e^{2x}\sin^2(y) = e^{2x}(\cos^2(y)+\sin^2(y)) = e^{2x}\cdot 1$

at the point (x_0, y_0), the Jacobian $\neq 0$ and is always positive. This means there is a continuous inverse of the mapping, defined in the open neighborhood of D^0 of (x_0, y_0). Intern the range is also an open neighborhood.

$\Rightarrow$ it is invertible in this neighborhood by the inverse function theorem.

b) The $\det(J) \neq 0$ on any (x,y) so the function is invertible in a domain. It is not invertible on the entire plane because it is not 1 to 1.

$F(0,0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$F(0,2\pi) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

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3) $f(x,y,z) = x^2+y^2+z^2 - C(xy+xz+yz)$

a) $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$

$\frac{\partial f}{\partial x} = 2x - C(y+z)$

$\frac{\partial f}{\partial y} = 2y - C(x+z)$ $\quad\quad \nabla f(0,0,0) = (0,0,0) = \vec{0}$

$-Cy-Cz \quad \frac{\partial f}{\partial z} = 2z - C(x+y)$

b) $H = \begin{pmatrix} 2 & -C & -C \\ -C & 2 & -C \\ -C & -C & 2 \end{pmatrix}$ $\quad (H-\lambda I) = 0 \Rightarrow \begin{pmatrix} 2-\lambda & -c & -c \\ -c & 2-\lambda & -c \\ -c & -c & 2-\lambda \end{pmatrix}$

$(2-\lambda)(2-\lambda)$
$(4-4\lambda+\lambda^2)(2-\lambda)$
$8-4\lambda-8\lambda+4\lambda^2+2\lambda^2-\lambda^3$

$\det(H)=(2-\lambda)\left((2-\lambda)^2-c\right)-(-c)\left(-c(2-\lambda)+c^2\right)+(-c)\left(c^2-(-c)(2-\lambda)\right)$

$=(2-\lambda)^3-c^2(2-\lambda)-(c^2(2-\lambda)+c^3)-(c^3+c^2(2-\lambda))$
$=-\lambda^3+6\lambda^2-12\lambda+8-\underline{c^2 2}+\underline{c^2\lambda}-\underline{2c^2}+\underline{c^2\lambda}-\underline{c^3}-c^3-\underline{2c^2}+\underline{c^2\lambda}$
$=-\lambda^3+6\lambda^2-12\lambda+8-2c^3+3c^2\lambda-6c^2$

Let $c=-1$
$-\lambda^3+6\lambda^2-12\lambda+8+2+3\lambda-6$
$=-\lambda^3+6\lambda^2-9\lambda-4 = (\lambda-1)(\lambda^2-5\lambda+4)$
$=(\lambda-1)(\lambda-1)(\lambda-4)$

$$\begin{array}{r}
\lambda^2-5\lambda+4 \\
\lambda-1 \overline{)\lambda^3-6\lambda^2+9\lambda-4} \\
\lambda^3+\lambda^2 \\
\hline
0-5\lambda^2+9\lambda \\
-5\lambda^2+5\lambda \\
\hline
0\ 4\lambda
\end{array}$$

Set = to 0, $\lambda=1,4$
since all $\lambda>0$, the CP $(0,0)$ is a relative minimum.

c) Take polynomial and let $c=2$
So:
$-\lambda^3+6\lambda^2-12\lambda+8-16+12\lambda-24$ -8
$=-\lambda^3+6\lambda^2-32=0$
Then $\lambda=-2$ & 4. This means this is not a relative minimum.

a)
4) $f(x,y)=e^x\sin(y)$ $\quad \vec{n}=\binom{1}{1}\frac{1}{\sqrt{2}}$

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$\nabla_{\vec{n}} f=\left(e^x\sin(y), e^x\cos(y)\right)\times\begin{pmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{pmatrix}$ → gradient

$=\dfrac{e^x\sin(y)}{\sqrt{2}}+\dfrac{e^x\cos(y)}{\sqrt{2}}$ This is the slope at (x,y)

b) $\Delta f=e^x\sin(y)-e^x\sin(y)=0$
$\hookrightarrow\left(\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}\right)$

5) $f: \mathbb{E}^n \to \mathbb{E}^1 \qquad f(x) = g(|x|)$

$g: \mathbb{E}^1 \to \mathbb{E}^1$

$\times 5$

Let $|x| = r \quad (r(\vec{x}) = |x|) \Rightarrow r(\vec{x}) = \sqrt{x_1^2 + x_2^2 + \dots x_n^2}$

$$\frac{\partial f}{\partial x_i} = g'(r) \cdot \frac{\partial r}{\partial x_i}$$

$$\frac{\partial^2 f}{\partial x_1^2} = g''(r) \cdot \cancel{\frac{\partial r}{\partial x_1} \cdot \frac{\partial r}{\partial x_1}} + g'(r) \cdot \frac{\partial^2 r}{\partial x_1^2} = g''(r) \cdot \frac{x_1^2}{x_1^2 + \dots x_n^2} + g'(r) \cdot \frac{(x_2^2 + \dots x_n^2)}{\sqrt{x_1^2 + \dots x_n^2} \cdot (x_1^2 + \dots x_n^2)}$$

nvm!

$$\frac{\partial r}{\partial x_1} = \frac{1}{2} \cdot \frac{1}{\sqrt{x_1^2 + \dots x_n^2}} \cdot 2x_1 = \frac{x_1}{\sqrt{x_1^2 \dots x_n^2}}$$

$$\frac{\partial^2 r}{\partial x_1^2} = \frac{(x_1^2 + \dots x_n^2)^{1/2} - x_1 \frac{x_1}{\sqrt{x_1 \dots}}}{(x_1^2 + x_2^2 + \dots x_n^2)} = \frac{(x_2^2 + \dots x_n^2)}{\sqrt{x_1^2 + \dots x_n^2} \cdot (x_1^2 + \dots x_n^2)}$$

In general $$\frac{\partial^2 f}{\partial x_k^2} = g''(r) \cdot \frac{x_k^2}{x_1^2 + \dots x_n^2} + g'(r) \cdot \frac{(x_1^2 + x_2^2 + \dots x_n^2) - x_k^2}{r \cdot (x_1^2 + \dots x_n^2)}$$

$\Delta f(x)$ is the sum of all $\frac{\partial^2 f}{\partial x_k^2}$ with $k = 1 \dots n$

$$\Rightarrow \Delta f(x) = g''(r) \sum_{k=1}^{n} \frac{x_k^2}{x_1^2 + \dots x_n^2} + g'(r) \sum_{k=1}^{n} \frac{(x_1^2 + x_2^2 + \dots x_n^2 - x_k^2)}{r \cdot (x_1^2 + x_n^2)}$$

$$\sum_{k=1}^{n} \frac{x_k^2}{x_1^2 + \dots x_n^2} = \frac{x_1^2}{x_1^2 + \dots x_n^2} + \frac{x_2^2}{x_1^2 + \dots x_n^2} + \dots \frac{x_n^2}{x_1^2 + x_n^2} = \frac{x_1^2 + x_2^2 + \dots x_n^2}{x_1^2 + x_2^2 + \dots x_n^2} = 1$$

$$\frac{1}{r(x_1^2 + \dots x_n^2)} \sum_{k=1}^{n} \frac{(x_1^2 + x_2^2 + \dots x_n^2 - x_k^2)}{1} = x_1^2(n-1) + x_2^2(n-1) \dots + x_n^2(n-1) \cdot \frac{1}{r(x_1^2 + x_n^2)}$$

$$= \frac{(n-1)}{r} \cdot \frac{x_1^2 + x_2^2 + \dots x_n^2}{x_1^2 + x_2^2 + \dots x_n^2} = \frac{n-1}{r}$$

So

$$\Delta f(x) = g''(r) + g'(r) \cdot \frac{n-1}{r} \qquad \square$$