21-241 Matrices and Linear Transformations Lecture 3
Midterm #2

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Section C

Time: 50 minutes

No textbook, calculator, recitation or exterior material is authorized. You can use your lecture notes but be aware that each time you go looking into them, you lose time. All statements should be justified. You have of course the right to use the results (theorems, remarks, examples...) provided in class. Distinct exercises are independent. Questions inside a fixed exercise might not be.

Question:	1	2	3	4	Total
Points:	10	15	10	15	50
Score:	10	15	10	15	50

1. Let $A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & 1 \\ 2 & 1 & -1 \end{bmatrix}$

   (a) (3 points) Show that A is invertible.

   (b) (7 points) Compute the inverse of A.

(a) $\det(A) = \begin{vmatrix} 1 & 1 & -1 \\ 1 & 0 & 1 \\ 2 & 1 & -1 \end{vmatrix} = -\begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} - \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = -(-1-2) - (1+1) = 3-2 = 1 \neq 0$

Thus, A is invertible.

(b) $[A|I_3] = \left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 2 & 1 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3-2R_1]{R_2-R_1} \left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 1 & 0 \\ 0 & -1 & 1 & -2 & 0 & 1 \end{array}\right]$

$\xrightarrow[-R_3]{R_3-R_2} \left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 1 & 0 \\ 0 & 0 & +1 & +1 & +1 & -1 \end{array}\right] \xrightarrow{R_1+R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 2 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & -1 \end{array}\right]$

$\xrightarrow[R_2-2R_3]{R_1-R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & 1 \\ 0 & -1 & 0 & -3 & -1 & 2 \\ 0 & 0 & 1 & 1 & 1 & -1 \end{array}\right] \xrightarrow{-R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & +3 & 1 & -2 \\ 0 & 0 & 1 & 1 & 1 & -1 \end{array}\right]$

Thus, $A^{-1} = \begin{bmatrix} -1 & 0 & 1 \\ 3 & 1 & -2 \\ 1 & 1 & -1 \end{bmatrix}$

2. We define

$$D_0 = 1, \quad D_1 = 0, \quad D_2 = \begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix}, \quad D_3 = \begin{vmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{vmatrix}, \quad D_4 = \begin{vmatrix} 0 & 1 & 1 & 1 \\ -1 & 0 & 1 & 1 \\ -1 & -1 & 0 & 1 \\ -1 & -1 & -1 & 0 \end{vmatrix}$$

More generally let D_n be the determinant of the $n \times n$ matrix with only 0 on the diagonal, only 1 above, and only -1 below :

$$D_n = \begin{vmatrix} 0 & 1 & \cdots & 1 \\ -1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ -1 & \cdots & -1 & 0 \end{vmatrix}$$

(a) (4 points) Compute D_2 and D_3.

(b) (4 points) Explain each step of the following computation:

$$D_4 = \begin{vmatrix} 0 & 1 & 1 & 1 \\ -1 & 0 & 1 & 1 \\ -1 & -1 & 0 & 1 \\ -1 & -1 & -1 & 0 \end{vmatrix} \underset{\text{step 1}}{=} \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & -1 & 0 & 1 \\ -1 & -1 & -1 & 0 \end{vmatrix} \overset{\text{step 2}}{=} \begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & -1 & 0 & 1 \\ -1 & -1 & -1 & 0 \end{vmatrix}$$

$$\overset{\text{step 3}}{=} -(-1) \begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & 0 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix} = D_2.$$

(c) (5 points) Prove that for all $n \geq 0$,

$$D_{n+2} = D_n.$$

Hint: spot D_n inside D_{n+2}.

(d) (2 points) Give without any proof or involved computation the value of $D_{1172016}$ and $D_{1172017}$.

(a) $D_2 = 1$. $D_3 = \begin{vmatrix} 1 & 1 \\ -1 & 0 \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1 - 1 = 0$ ✓

(b) step 1. Add the 4th col to the 1st col and det doesn't change.

step 2. Add the 4th row to the 1st row and ~~er~~ det doesn't change. ✓

step 3 Do cofactor expansion along the 1st col, which only has one nonzero ~~term~~ entry. (the last one).

step 4. Do cofactor expansion along the 1st row, which only has one nonzero entry (the last one).

Let $n \in \mathbb{N}$.

(c) $D_{n+2} = \begin{vmatrix} 0 & 1 & \cdots & 1 \\ -1 & \ddots & & \vdots \\ \vdots & & \ddots & 1 \\ -1 & \cdots & -1 & 0 \end{vmatrix} \underset{C_1+C_n}{=} \begin{vmatrix} 1 & 1 & \cdots & 1 \\ 0 & & & \vdots \\ \vdots & & & \\ 0 & & & 1 \\ -1 & -1 & \cdots -1 & 0 \end{vmatrix} \underset{R_1+R_n}{=} \begin{vmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & & & \vdots \\ \vdots & & & & 1 \\ 0 & & & & \\ -1 & -1 & \cdots & -1 & 0 \end{vmatrix}$

If n is even, $D_{n+2} = -(-1) \begin{vmatrix} 0 & 0 \cdots & 0 & 1 \\ 0 & 1 \cdots & & 1 \\ \vdots & & & \vdots \\ -1 & \cdots & 0 & 1 \end{vmatrix} = 1 \cdot \begin{vmatrix} 0 & 1 & \cdots & 1 \\ -1 & & & \vdots \\ \vdots & & & 1 \\ -1 & \cdots & -1 & 0 \end{vmatrix} = D_n$

good!

If n is odd, $D_{n+2} = (-1) \begin{vmatrix} 0 & 0 \cdots & 0 & 1 \\ 0 & 1 \cdots & & 1 \\ \vdots & & & \vdots \\ -1 & \cdots & 0 & 1 \end{vmatrix} = -1 \cdot (-1) \begin{vmatrix} 0 & 1 & \cdots & 1 \\ -1 & & & \vdots \\ \vdots & & & 1 \\ -1 & \cdots & -1 & 0 \end{vmatrix} = D_n$ □

(d) $D_{1172016} = 1$. $D_{1172017} = 0$ ✓

3. Let

$$B = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & 1 \\ -1 & 1 & -1 \end{bmatrix}$$

(a) (4 points) Is 0 an eigenvalue of B ?

(b) (2 points) What is the rank of B ?

(c) (2 points) Give without doing any computation (but still justifying your answer !) what the rowspace and columspace of B are.

(d) (2 points) What is the dimension of the rowspace of

$$C = \begin{bmatrix} 1 & 0 & 1 & \pi \\ 2 & -1 & 1 & e \\ -1 & 1 & -1 & \sqrt{2} \end{bmatrix}$$

and of which bigger vector space is it a subspace of ?

(a) $|B-\lambda I| = \begin{vmatrix} 1-\lambda & 0 & 1 \\ 2 & -1-\lambda & 1 \\ -1 & 1 & -1-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} -1-\lambda & 1 \\ 1 & -1-\lambda \end{vmatrix} + \begin{vmatrix} 2 & -1-\lambda \\ -1 & 1 \end{vmatrix} = (1-\lambda)[(\lambda+1)^2-1) + 2-(\lambda+1)$

$= (1-\lambda)(\lambda^2+2\lambda) - \lambda + 1$

Let $\lambda = 0$: $|B-\lambda I| = 1 \neq 0$. Thus, 0 is not an eigenvalue of B ✓

(b) rank$(B) = 3$. Because 0 is not an eigenvalue of B is equivalent to ran$(B) = 3$. ✓

(c) By the fundamental theorem of invertible matrices, 0 is not an eigenvalue of B is equivalent to the row vectors of B span $\mathbb{R}^n$ and also equivalent to the column vectors of B span $\mathbb{R}^n$.

Thus, the rowspace and columnspace of B are both $\mathbb{R}^3$. ✓

(d) $C = \begin{bmatrix} 1 & 0 & 1 & \pi \\ 2 & -1 & 1 & e \\ -1 & 1 & -1 & \sqrt{2} \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & \pi \\ 0 & -1 & -1 & e-2\pi \\ 0 & 1 & 0 & \sqrt{2}+\pi \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & \pi \\ 0 & -1 & -1 & e-2\pi \\ 0 & 0 & -1 & e+\sqrt{2}-\pi \end{bmatrix}$

Thus, dim row$(C) = 3$ and row(C) is a subspace of $\mathbb{R}^4$. ✓

4. Let

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

(a) (6 points) Find the eigenvalues of A.

(b) (1 point) Is A invertible ?

(c) (4 points) Compute the eigenspace associated to the eigenvalue with algebraic multiplicity 2.

(d) (2 points) Is A diagonalizable ?

(e) (2 points) Compute $A^{2017}\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$.

(a) $A-\lambda I = \begin{bmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{bmatrix}$

$\det(A-\lambda I) = -\lambda\begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 1 & -\lambda \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ -\lambda & 1 \end{vmatrix}$

$= -\lambda(\lambda^2-1) - (-\lambda-1) + 1+\lambda$

$= -\lambda^3 + \lambda + \lambda + 1 + 1 + \lambda$

$= -\lambda^3 + 3\lambda + 2 = -(\lambda-2)(\lambda^2+2\lambda+1) = -(\lambda-2)(\lambda+1)^2 \Rightarrow \lambda_1 = 2 \; \lambda_2 = -1$

(b) Since 0 is not an eigenvalue of A, A is invertible

(c) ~~$\lambda_1 = 2: \quad A-2I = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -2 \\ 1 & -2 & 1 \\ -2 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -2 \\ 0 & -3 & 3 \\ 0 & 3 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$~~

~~$x_1 = x_3 \quad x_2 = x_3 \quad E_2 = \text{span}\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right\}$~~

$\lambda_2 = -1: \quad A+I = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad x_1 = -x_2 - x_3 \quad E_{-1} = \left\{\begin{bmatrix} -x_2-x_3 \\ x_2 \\ x_3 \end{bmatrix}, x_2, x_3 \in \mathbb{R}\right\}$

$= \text{span}\left\{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right\}$

(d) Yes. Because for each eigenvalue, its geometric multiplicity ~~must~~ $\leq$ algebraic multiplicity.
~~Thus~~, Since $\lambda_1 = 2$ has algebraic multiplicity 1, its geometric multiplicity must also be 1.
By (c), $\lambda_2 = -1$ has algebraic multiplicity and geometric multiplicity both equal to 2.
Thus, the algebraic multiplicity of each eigenvalue of A ~~is~~ equals its geometric multiplicity.
Therefore, A is diagonalizable.

(e) $A^{2017}\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} = A^{2017}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + A^{2017}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$= (-1)^{2017}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + (-1)^{2017}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$= \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}$