21-241 Matrices and Linear Transformations Lecture 3 Final

December 12, 2016

Time: 3 hours

Only your 5 over 6 best exercises will count towards your grade over 100. Half of the remaining credit will be added as bonus points.

No textbook, calculator, recitation or exterior material is authorized. You can use your lecture notes. All statements should be justified. You have of course the right to use the results (theorems, remarks, examples...) provided in class. Distinct exercises are independent. Questions inside a fixed exercise might not be.

Question:	1	2	3	4	5	6	Total
Points:	20	20	20	20	20	20	120
Score:							

1. Answer by yes with a short but precise justification or no with a counterexample.
   (a) (2 points) If $v_1, \cdots v_n \in \mathbb{R}^n$ are independent, then every equation $[v_1|\cdots|v_n]\, x = b$ has always a unique solution.
   (b) (3 points) If a matrix is invertible, then it is diagonalizable.
   (c) (2 points) For every equation $Ax = b$ with $A \in \mathcal{M}_{nn}(\mathbb{R})$ and $b \in \mathbb{R}^n$, the set of solutions is a subspace of $\mathbb{R}^n$.
   (d) (3 points) The only diagonalizable nilpotent matrix is 0 *(a matrix M is called nilpotent if there exists an integer k such that $M^k = 0$).*
   (e) (3 points) If A is $m \times n$ with $m < n$ then for every $b \in \mathbb{R}^m$ the equation $Ax = b$ has a solution.
   (f) (3 points) If A, B are $n \times n$ matrices, $\det(A + B) = \det(A) + \det(B)$
   (g) (4 points) For $a, b \in \mathbb{C}$ and $A, B \in \mathcal{M}_{nn}(\mathbb{C})$, the following diagonal block product formula holds

$$\left[\begin{array}{c|c} a & 0\cdots 0 \\ \hline 0 & \\ \vdots & \Large A \\ 0 & \end{array}\right]\left[\begin{array}{c|c} b & 0\cdots 0 \\ \hline 0 & \\ \vdots & \Large B \\ 0 & \end{array}\right] = \left[\begin{array}{c|c} ab & 0\cdots 0 \\ \hline 0 & \\ \vdots & \Large AB \\ 0 & \end{array}\right].$$

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2. Let $n \geq 1$ and $\mathrm{tr} : \mathcal{M}_{nn}(\mathbb{R}) \to \mathbb{R}$ defined by

$$\mathrm{tr}(A) = \sum_{i=1}^{n} a_{ii}$$

   (a) (3 points) Show that tr is a linear transformation.
   (b) (5 points) Show that $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ for all matrices $A, B \in \mathcal{M}_{nn}(\mathbb{R})$.
   (c) (5 points) Is it possible that there exists two $n \times n$ matrices A, B such that $AB - BA = I_n$?
   (d) (7 points) For a diagonalizable A, show that $\mathrm{tr}(A)$ is the sum of the eigenvalues of A.
   Remark for fun: using Exercise 5 one can show that this is actually true for all matrices.

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3. Let $v_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ and $v_2 = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$.
   (a) (3 points) Find a unit vector v_3 orthogonal to v_1 and v_2.
   (b) (3 points) Why is $C := (v_1, v_2, v_3)$ a basis of $\mathbb{R}^3$?
   (c) (3 points) Let $v = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \mathbb{R}^3$. What are the coordinates of v in C ?
   (d) (3 points) Let B be the standard basis of $\mathbb{R}^3$. Write down the change of basis matrix $P_{C \leftarrow B}$.
   (e) (4 points) Let p be the orthogonal projection onto $\mathrm{span}(v_1, v_2)$. What is $[p]_C$ the matrix of p with bases C to C ?
   (f) (4 points) Compute $[p]_B$.

4. Let $a, b, c \in \mathbb{R}$.
   (a) (10 points) Show that the determinant
$$D(a,b,c) := \begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} = abc(b-a)(c-a)(c-b)$$
   (b) (10 points) Deduce the value of
$$E(a,b,c) := \begin{vmatrix} a+b & b+c & c+a \\ a^2+b^2 & b^2+c^2 & c^2+a^2 \\ a^3+b^3 & b^3+c^3 & c^3+a^3 \end{vmatrix}$$

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5. Let $W = \operatorname{span}\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right)$
   (a) (5 points) What is the orthogonal complement of W ?
   (b) (3 points) Let $A = \frac{1}{3}\begin{bmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$. Compute $A\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $A\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ and $Aw^{\perp}$ for some $w^{\perp} \in W^{\perp}$ of your choice.
   (c) (8 points) Orthogonally diagonalize A (that is, find an orthogonal Q and a diagonal D such that $Q^T AQ = D$).
   (d) (4 points) Characterize the geometrical transformation underlying A and draw a picture.

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6. The aim of this exercise is to show that any $A \in \mathcal{M}_{nn}(\mathbb{C})$ is similar to a triangular matrix, i.e., there exists $P \in \mathcal{M}_{nn}(\mathbb{C})$ invertible and $T \in \mathcal{M}_{nn}(\mathbb{C})$ upper triangular (i.e. a matrix with only zeros below the diagonal) such that
$$A = PTP^{-1}.$$
   (a) (3 points) Prove the result when $n = 1$.
   (b) (3 points) Assume the result up to size $n-1$ for some integer $n \geq 2$ and pick $A \in \mathcal{M}_{nn}(\mathbb{C})$. What theorem enables you to say that the characteristic polynomial of A has at least one root $\lambda_1 \in \mathbb{C}$?
   (c) (6 points) Call v_1 an eigenvector associated to λ_1. Explain why A is similar to some
$$\left[\begin{array}{c|c} \lambda_1 & L \\ \hline 0 & \\ \vdots & B \\ 0 & \end{array}\right]$$
   where $L \in \mathcal{M}_{1,n-1}(\mathbb{C})$ and $B \in \mathcal{M}_{n-1,n-1}(\mathbb{C})$ are some unknown matrices.
   Hint: take inspiration from the proof of the spectral theorem by doing an appropriate change of basis.
   (d) (5 points) Conclude by constructing suitable P and T.
   Hint: take inspiration from the proof of the spectral theorem by doing a second appropriate change of basis and put the two together.
   (e) (3 points) What is wrong if one replaces everywhere $\mathbb{C}$ by $\mathbb{R}$?