Matrices and Linear Transformations

# Homework 8

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## Excercise 1

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1. If $AB$ is invertible, then there exists some matrix $(AB)^{-1}$ such that
$(AB)^{-1}AB = I_n$
$((AB^{-1})A)B = I_n$, so $(AB^{-1})A = B^{-1}$ and $B$ is invertible ✓
since this is a matrix that can be multiplied by $B$ to get $I_n$.
Same for A:
$AB(AB^{-1}) = I_n$ if $AB$ is invertible
$A(B(AB)^{-1}) = I_n$ shows that $A^{-1}$ exists and $A^{-1} = (B(AB)^{-1})$
Therefore, if $AB$ is invertible, both $A$ and $B$ must be invertible.

2 By contraposition of the statement proved in (1), if either $A$ or $B$ are not invertible, then $AB$ is not invertible. ✓

## Excercise 2

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1. $\det(B^m) = \det(B)^m$
If $B^m = 0$, then $\det(0) = 0 = \det(B)^m$ $\qquad 0 = x^m$
$\det(B) = 0$ ✓

2. $\det(A^2) = \det(A)^2 = \det(A)$ if $A^2 = A$ $\qquad x^2 = x$
$\det(A)$ could be $0$ or $1$ ✓

## Excercise 3

4

A matrix is invertible iff its determinant $\neq 0$

$$\det(A) = k\begin{vmatrix} 2 & k \\ k & k \end{vmatrix} - k\begin{vmatrix} k^2 & k \\ 0 & k \end{vmatrix} + 0$$

$\det(A) = k(2k - k^2) - k(k^3 - 0) = -k^3 + 2k^2 - k^4 \neq 0$
$-k^2(k^2 + k - 2) = -k^2(k-1)(k+2) = 0$
A will be invertible for all $k \in \mathbb{R}$ except $k = 0$, $k = 1$, $k = -2$. ✓

Excercise 4

1. $$\det(A-\lambda I_n) = \begin{vmatrix} 1-\lambda & 0 & 1 \\ 0 & 1-\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = (1-\lambda)(-\lambda(1-\lambda)-1) - 0 + 1(0-(1-\lambda))$$

$$\det(A-\lambda I_n) = -\lambda^3 + 2\lambda^2 + \lambda - 2 = -(\lambda-2)(\lambda-1)(\lambda+1) = 0$$

$$\boxed{\lambda = 1, \ \lambda = -1, \ \lambda = 2}$$ ✓

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2. $\lambda = 1$:

$$A - I_n = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{aligned} x &= -y \\ y &= 1 \\ z &= 0 \end{aligned}$$

$$E_1 = \text{span}\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$$ ✓

$\lambda = -1$:

$$A + I_n = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{aligned} x &= -\tfrac{1}{2}z \\ y &= -\tfrac{1}{2}z \\ z &= -2 \end{aligned}$$

$$E_{-1} = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \right\}$$ ✓

$\lambda = 2$:

$$A - 2I_n = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 1 & 1 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{aligned} x &= z \\ y &= z \\ z &= 1 \end{aligned}$$

$$E_2 = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$ ✓