21-241 – Homework assignment week #6

Laurent Dietrich
Carnegie Mellon University, Fall 2016, Sec. F and G

Reminder

Homework will be given on Fridays and due on the next Friday before 5pm, to me in class or in Andrew Zucker's mailbox in Wean Hall 6113. Late homework will never be accepted without a proper reason. In case of physical absence, electronic submissions by e-mail to both me and Andrew Zucker can be accepted. Please do not forget to write your name, andrew id and section and please use a staple if you have several sheets.

Reading

- Poole: Sec. 3.3

Problem (19 pts)

The aim of this homework is to work on the product formula

$$(AB)_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

and on the concept of inverse of a matrix. A very good problem to train on both of these is to prove the proposition I stated in class : *any elementary matrix is invertible and its inverse is the elementary matrix associated to the reverse elementary operation.*

1. Standard basis of $\mathcal{M}_{nn}(\mathbb{R})$ (8 pts)

Let $n \geq 1$ and $1 \leq i, j, k, l \leq n$. We call E_{ij} the matrix with a 1 only at entry (i, j) and 0 elsewhere. Observe that the E_{ij} form the standard basis of $\mathcal{M}_{nn}(\mathbb{R})$ in the sense that for any matrix $A = ((a_{ij})) \in \mathcal{M}_{nn}(\mathbb{R})$ one has

$$A = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} E_{ij}$$

just as the e_i were forming the standard basis for vectors, where we had only one sum.

The goal of this exercise is to prove the product formula

$$E_{ij} E_{kl} = \delta_{jk} E_{il} \tag{1}$$

where δ_{jk} is the Kronecker symbol, that is $\delta_{jk} = \begin{cases} 1 \text{ iff } j = k \\ 0 \text{ otherwise} \end{cases}$.

1. (3 pts) Pick $1 \leq r, s \leq n$ and prove that $(E_{ij} E_{kl})_{rs} = 0$ if $r \neq i$ or $s \neq l$. Be careful not to use i, j, k, l, r or s as a summation index !

2. (2 pts) What is $(E_{ij} E_{kl})_{il}$ (depending on j and k of course) ?

3. (3 pts) Conclude the proof of formula (1).

2. Elementary matrix inverse (11 pts)

For all $1 \leq i, j \leq n$ and $\lambda \in \mathbb{R}$ we define

$$A = \begin{bmatrix} 1 & & & & & & & \\ & \ddots & & & & & & \\ & & 1 & & & & & \\ & & & \lambda & & & & \\ & & & & 1 & & & \\ & & & & & \ddots & & \\ & & & & & & 1 \end{bmatrix} = I_n + (\lambda - 1)E_{ii}$$

$$B = \begin{bmatrix} 1 & & & & & & & \\ & \ddots & & & & & & \\ & & 1 & & & & & \\ & & & \ddots & & & & \\ & & \lambda & & 1 & & & \\ & & & & & \ddots & & \\ & & & & & & 1 \end{bmatrix} = I_n + \lambda E_{ij}$$

$$C = \begin{bmatrix} 1 & & & & & & & \\ & \ddots & & & & & & \\ & & 0 & & 1 & & & \\ & & & \ddots & & & & \\ & & 1 & & 0 & & & \\ & & & & & \ddots & & \\ & & & & & & 1 \end{bmatrix} = ?$$

where C is I_n with rows i and j exchanged.

1. (2 pts) Propose a formula for C involving I_n and some matrices from the standard basis.
2. (3 pts) Explain to which elementary row operations are A, B and C corresponding.
3. (4 pts) Prove that A is invertible when $\lambda \neq 0$ by finding a matrix A' such that $AA' = I_n$. You will express this inverse candidate with I_n and some matrices from the standard basis, and use formula (1).
4. (2 pts) Same question for B and C.