Concepts Sample Final

Solutions can be found at http://www.andrew.cmu.edu/user/suesugi/concepts/samplefinalsol/

1. Let x, y be the real numbers. Find the minimum value of the following expression.

$$x^2 + y^2 - 8x - 5y + 15$$

2. Assume $sin(x) \neq 0$. Prove the following for all natural number *n*.

$$\prod_{i=0}^{n-1} \cos(2^{i} x) = \frac{\sin(2^{n} x)}{2^{n} (\sin(x))}$$

Hint: sin(2x) = 2sin(x)cos(x) for any *x*.

3. Prove that for all natural number n,

$$x + \frac{1}{x} \in \mathbb{Z} \Longrightarrow x^n + \frac{1}{x^n} \in \mathbb{Z}$$

4. Is the following function surjective if $f : \mathbb{R} \to \mathbb{R}$? Explain.

$$f(x) = e^{3x^2 + 2} + 1$$

Is the following function injective if $f : \mathbb{R} \to \mathbb{R}$? Explain.

$$f(x) = e^{3x^2 + 2} + 1$$

5. Prove by counting 2 ways:

$$\sum_{k=0}^{n} k \binom{n}{k} = n2^{n-1}$$

6. Let n be a natural number greater than or equal to 3. Prove that

 $n-2 \mid 2n$

If and only if n = 3, 4, 6.

7. There are three closed boxes on the table. The First Box has 2 red and 3 black balls, the Second Box has 3 red and 2 black balls and the Third Box has 5 red balls. A man rolls a die. If the result is 1, he opens the First Box; if the result is even, he opens the Second Box; if the result is either 3 or 5, he opens the Third Box. After that he picks a ball from the open box. What is the probability that the other four balls in the box are red given that the selected ball is red?

8. Suppose we have 100 students in the class. What is the expected number of **pairs** of students with the same birthday? Note that there are 366 distinct birthdays.