MAPLE FINAL

This exam is due at midnight on Tuesday May 16. You may work on it in any 48-hour period of your choice; please record your starting and finishing times. Please consult with me if anything is unclear. You should attempt exactly five questions. Each question is worth 20 points.

1. Let $n > 2$ be an odd composite integer, and let $n - 1 = 2^s t$ where s and t are integers with t odd (notice that such s and t always exist and are uniquely determined by n).

   Let b be an integer. n is called a *strong pseudoprime to the base* b if

   (a) EITHER b^t is congruent to 1 modulo n

   (b) OR there exists an integer r with $0 \leq r < s$ such that $b^{2^r t}$ is congruent to -1 modulo n.

   The reason for this terminology is that if n was prime, it would have this property for every integer b with $0 < b < n$.

   Let $f(n)$ be the number of b such that $0 < b < n$ and n is a strong pseudoprime to the base b. Write a Maple program which takes input n and returns $f(n)$.

   Now generate 100 random composite integers less than 10000 and compute $f(n)$ for each such integer. Make a conjecture about the maximum possible size of $f(n)$ compared to n when n is composite.

   How could we use strong pseudoprimes in a probabilistic primality test?

2. Let $f(x, y, z)$ be some function defined on 3-dimensional space where (x, y, z) are the usual Cartesian coordinates. Consider the differential equation

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$$

   Suppose that instead of thinking of f as a function of (x, y, z) we want to work in spherical coordinates, where we describe a point in terms of its distance r from the origin and two angles θ

and ϕ. The translation is given by

$$x = r\cos(\theta)\sin(\phi), y = r\sin(\theta)\sin(\phi), z = r\cos(\phi).$$

Use Maple to

(a) Write the equation in spherical coordinates. Hint: look in the online help under ?laplacian and ?coords.

(b) Find a non-zero solution of the form $R(r)\Theta(\theta)\Phi(\phi)$.

3. Let f be a function of t. The differential equation

$$\frac{df}{dt} = f(1-f)$$

is often used by ecologists to model the growth of animal populations, where $f(t)$ is the population at time t.

(a) Find the general solution of the equation.

(b) For a real number a let $f_a(t)$ be the solution of the equation which obeys the initial condition $f_a(0) = a$. Plot a single graph which shows the behaviour of

$$\{f_0, f_{0.25}, f_{0.5}, f_{0.75}, f_1 . f_{1.25}, f_{1.5}, f_{1.75}, f_2\}$$

for t in the range 0 to 10.

(c) Comment on the biological interpretation of your graph.

4. Let a function θ be given by

$$\theta(t) = \sum_{n=-\infty}^{\infty} e^{-\pi t n^2}$$

for all $t > 0$.

(a) Write a Maple program which computes a value for $\theta(t)$ that is accurate to within 10^{-10}.

(b) Draw a graph that shows the behaviour of $\theta(t)$ in the interval $[0, 4]$.

(c) Draw a graph that shows the behaviour of $\theta(t) - t^{-1/2}$ in $[0, 4]$.

(d) What do the graphs tell you about the behaviour of $\theta(t)$ for t small and positive?

5. Consider the differential equation

$$\frac{d^2x}{dt^2} + x = \cos(\lambda t)/10$$

where λ is a numerical parameter.

(a) Use Maple to find the general solution.

(b) For which value(s) of λ do you suspect that your result may be wrong?

(c) Use Maple to solve the equation for the suspicious value(s) of λ.

(d) Draw the solutions of the equation with initial conditions $x(0) = 0, x'(0) = 1$ for $\lambda = 0.5, 0.7, 0.99, 1, 1.01, 1.3, 1.5$. Comment on your results.

6. Let c be a complex number. Define a sequence $z_0, z_1, z_2 \ldots$ of complex numbers by setting $z_0 = 0$ and $z_{n+1} = z_n^2 + c$ for all n. The *Mandelbrot set* is defined to be the set of points c such that $|z_n|$ is bounded as n goes to infinity.

Write a program that takes c as input, computes the first 100 members of the sequence z_n and then returns a reasonable guess at whether c is in the Mandelbrot set or not.

Hint: first write a program to compute the z_n, then play with it and see how the sequence of z_n behaves as c varies.

7. Let p be an odd prime and let a be an integer not divisible by p. We say that a is a *quadratic residue modulo* p when there is b such that b^2 is congruent to a modulo p; for example 11 is a quadratic residue modulo 5 but 12 is not.

Let a, p be as above and let $r = a^{(p-1)/2} \mod p$. A remarkable theorem says that if a is a quadratic residue mod p then $r = 1$, and if a is not a quadratic residue then $r = p - 1$. To put it another way $a^{(p-1)/2}$ is always congruent to $+1$ or -1 and the sign is determined by whether or not a is a quadratic residue mod p.

(a) Use Maple to create a table which tells you, for each pair of odd primes (p, q) both less than 100, whether p is a quadratic residue modulo q and whether q is a quadratic residue modulo p.

(b) Use the data from your table to make a conjecture about the relation between the statements "p is a residue mod q" and "q is a residue mod p" for odd primes p and q.

(c) Generate two random 125-digir primes p and q and test the truth of your conjecture.