MATH STUDIES ALGEBRA SPRING 2018: HOMEWORK 3

JC

This homework is due by class time on Monday 12 February. It must be typeset (preferably in LaTeX) and submitted as a PDF file on the Canvas site, with a filename of the form

`andrewID_alg_homeworknumber.pdf`

For each minute that it is late, the grade will be reduced by 10 percent.

(1) Let V be a VS over a field k. Prove that:
  (a) A subset X of V is a basis of V if and only if it is a *minimal spanning set*, that is to say X is spanning but every proper subset $Y \subsetneq X$ fails to be spanning.
  (b) If X is spanning and Y is a maximal element in the poset of independent subsets of X (ordered by inclusion) then Y is a basis.
  (c) If X is independent and Y is spanning then $|X| \leq |Y|$.

(2) Prove that if W is a VS over k and V is a subspace of W then for any basis B of V there is a basis C of W with $B \subseteq C$. Prove further that in this case the set $\{c + V : c \in C \setminus B\}$ is a basis for the quotient space W/V. Use this to deduce that $\dim(W) = \dim(V) + \dim(W/V)$.

(3) Let $0 \to V_0 \to \ldots \to V_n \to 0$ be an exact sequence of VS's over a field k (so the maps from V_i to V_{i+1} are linear for $i < n$). Assume that $\dim(V_i)$ is finite for each i. Prove that $\sum_{i=0}^{n} (-1)^i \dim(V_i) = 0$.

(4) Let $f : \mathbb{R} \to \mathbb{R}$ be such that $f(x + y) = f(x) + f(y)$ and $f(1) = 1$.
  (a) Prove that $f(q) = q$ for all rational q.
  (b) Prove that if f is continuous then f is the identity function.
  (c) Show that there exists an f as above which is not the identity function. Hint: $\mathbb{R}$ can be viewed as a VS over $\mathbb{Q}$.

(5) Let P be a nonempty set of prime numbers. Let S be a set of integers such that for each $k \in S$, $k > 1$ and only primes in P appear in the prime factorisation of k. Prove that if $|S| > |P|$ then there is a nonempty subset S' of S such that the product of the elements of S is a perfect square. Hint: Make it into a problem about linear independence in the k-VS $k^{|P|}$ where $k = \mathbb{Z}/2\mathbb{Z}$.