21-320 Takehome Midterm

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Due at midnight on Sunday 5 March. No collaboration allowed. Feel free to consult with me if you are having problems. I encourage submission by email (preferably as a PostScript file) to `maplehw@asdb.math.cmu.edu`; make sure to put your name in the file that you send me, preferably at the start!

1. Let a curve C be given by $y^2 = x^3 - 2$.

   (a) Plot the curve C in the region where $0 \leq x \leq 4$ and $-8 \leq y \leq 8$. Are there any points on the curve C which lie to the left of the y-axis?

   (b) Find an expression for the slope of the tangent line to C at a point (x, y) lying on C. Hint: differentiate implicitly and solve for y'.

   (c) Given a point $P = (x_P, y_P)$ on the curve C with $y_P \neq 0$, find

      i. The equation of the tangent line to the curve C at the point P.

      ii. The other point on C where the tangent line to C at P intersects C.

   Your answers should be expressed as arrow functions which take x_P and y_P as arguments, and return the equation of the tangent and the coordinates of the desired point respectively.

   Why do you think I stipulated that $y_P \neq 0$?

   (d) Check your answer to the last part by redrawing the graph of C and adding the tangent line at some specific point P.

   (e) Given points $P = (x_P, y_P)$ and $Q = (x_Q, y_Q)$ on the curve C with $x_P \neq x_Q$, find

i. The equation of the unique straight line passing through P and Q.

ii. The third point on C where the straight line through P and Q intersects C.

As before your results should be exprssed as arrow functions.

(f) Check your answer to the last part by redrawing the graph of C and adding the line through some specific pair of points P, Q.

(g) Given P and Q as in the last part, let $R = (x_R, y_R)$ be the third point where the line through P and Q intersects C; we define $P + Q$ to be the point $(x_R, -y_R)$. Prove that $P + Q$ is a point of C.

Find three specific points P_0, P_1, P_2 on C such that all the expressions $P_0 + P_1$, $P_1 + P_2$, $P_0 + (P_1 + P_2)$, $(P_0 + P_1) + P_2$ are well-defined. Evaluate them. What do you notice?

2. Consider the differential equation $y^2 + \left(\frac{dy}{dx}\right)^2 = 1$ with boundary condition $y(0) = 1$.

   (a) Solve the equation analytically using "dsolve".

   (b) Find a solution which Maple did not find.

3. Give an expansion of $\ln(x)$ in powers of $x - 2$ up to the term of order 10. Find the range of values of x in which this expansion differs from the true value by at most 10^{-3}.

4. Consider the second order differential equation

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - 1)y = 0,$$

with the boundary conditions $y(0) = 0$, $y'(0) = 1$.

   (a) Use "dsolve" to solve the equation exactly. Plot the resulting solution for x in $[-10, +10]$.

(b) Find an approximate series solution in powers of x, involving powers of x up to x^3. Plot the series solution together with the exact solution from the previous part in the interval $[-10, 10]$. Over what range of values is the series solution within 0.01 of the exact solution?

(c) How many terms do you need to take in a series solution to get one which is within 0.01 of the exact solution throughout the interval [-10,+10]?

5. For this question, you should use the fact that the Taylor expansion of $\tan^{-1}(x)$ is

$$x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$$

and that this power series converges to $\tan^{-1}(x)$ for all x.

(a) Since $\tan^{-1}(1) = \pi/4$, we can compute $\pi/4$ by summing the series

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

Roughly how many terms would we need to take to get a value for $\pi/4$ accurate to ten decimal places?

(b) Find an expression for $\tan(x + y)$ in terms of $\tan(x)$ and $\tan(y)$.

(c) Prove that $\pi/4 = 4\tan^{-1}(1/5) - \tan^{-1}(1/239)$. Hint: take the tan of the RHS using the formula you just found.

(d) Roughly how many terms of the expansion

$$\left(\frac{4}{5} - \frac{1}{239}\right) - \left(4\left(\frac{1}{5}\right)^3 - \left(\frac{1}{239}\right)^3\right) + \left(4\left(\frac{1}{5}\right)^5 - \left(\frac{1}{239}\right)^5\right) - \dots$$

should we take to get a value for $\pi/4$ which accurate to ten decimal places?