# SKETCHY NOTES FOR WEEKS 9 AND 10, PART TWO

## 1. Examples with graphs

The language of graph theory has equality and one binary relation symbol $E$ (edge). The axioms of graph theory say that $\forall x \neg xEx$ and $\forall x \forall y\ (xEy \implies yEx)$, call the theory with these two sentences $T_g$. We usually think of a graph as a pair $G = (V_G, E_G)$ where $E_G$ is a set of unordered pairs from $V_G$. Element of $V_G$ are *vertices*, elements of $E_G$ are *edges*. Some simple properties of graphs can be expressed by sentences, for example “every vertex has three neighbours”.

A graph is *connected* if for any two distinct vertices $v, w$ there is a sequence $v_0 = v, \dots v_n = w$ such that $v_i E v_{i+1}$ for all $i < n$. This property can’t be expressed in FOL, even if we expand the langauge and allow infinitely many sentences. Intuitively, this is because we would need an infinite disjunction of increasingly complicated formulae.

**Theorem 1.** *There is no expansion of the language of graphs and no theory $T$ in this expansion such that the set of reducts to the language of graph theory of models of $T$ is exactly the set of connected graphs.*

*Proof.* Consider the expansion of the langauge of $T$ by two new constants $c$ and $d$, and let $\phi_n$ express “there is no chain of edges of length $n$ which joins $c$ to $d$”. We claim the theory $T \cup \{\phi_n : n > 0\}$ has a model, for which (Compactness) it’s enough to show that $T \cup \{\phi_n : 0 < n < N\}$ has a model for all $N$.

Let $G$ be a “linear” graph with $N$ edges. $G$ is connected so we can take an expansion to the langauge of $T$ and get a model of $T$, then we can expand further by interpreting $c$ and $d$ as the endpoints and get a model of $T \cup \{\phi_n : 0 < n < N\}$.

Now take a model of $T \cup \{\phi_n : n > 0\}$. Its reduct to the language of graphs is not a connected graph because the interpretations of $c$ and $d$ can’t be joined. $\square$

An *$n$-cycle* in a graph is a sequence $v_1, \dots v_n$ of distinct vertices such that $v_i$ and $v_{i+1}$ are joined by an edge, for $i < n$, as are $v_n$ and $v_1$. A graph is *acyclic* if it has no $n$-cycles for $n \geq 3$. Clearly we can find an infinite theory $T_a$ in the language of graphs whose models are the acyclic graphs: $T_a$ is the union of $T_g$ and the set $\{\psi_n : n \geq 3\}$ where $\psi_n$ expresses “there is no $n$-cycle”.

**Theorem 2.** *There is no expansion of the language of graphs and no finite theory $T$ in this expansion such that the set of reducts to the language of graph theory of models of $T$ is exactly the set of acyclic graphs.*

*Proof.* Clearly $T_a \models T$, so there is some $N$ such that $T_g \cup \{\psi_n : 3 \leq n \leq N\} \models T$. Now take an $(N+1)$-cycle, this is a model of the LHS and if we expand it to the language of $T$ we get a model of the RHS, contradiction. $\square$

A *$k$-colouring* of a graph $G$ is a function $f : V_G \to \{1, \dots k\}$ such that $f(v) = f(w) \implies \neg vEw$ for all $v, w$. We can express the property of being $k$-colourable by expanding the language with $k$ unary predicates and writing down a long sentence $\psi_k$: $\psi_k$ says that every element satisfies exactly one $R_i$, and that for each $i$ there do not exist neighbouring elements satisfying $R_i$. The $k$-colourable graphs are exactly the models of $T_g \cup \{\psi_k\}$.

If $W \subseteq V_G$, the *induced subgraph of $G$ induced by $W$* is $(W, E \cap [W]^2)$.

**Theorem 3.** *A graph is $k$-colourable if and only if every finite induced subgraph is $k$-colourable.*

*Proof.* Consider the theory which is the union of the atomic diagram of $G$ and $T_g \cup \{\psi_k\}$. Every finite subset is consistent, so the whole theory is consistent. From a model of the whole theory (which is graph containing an isomorphic copy of $G$, and having a $k$-colouring) we read off a $k$-colouring of $G$. $\square$

## 2. Dense linear orderings without endpoints

The language of linear orderings has a relation symbol $\leq$. A linear ordering is dense if $x < y$ implies there is $z$ with $x < z < y$. DLOWE is the (finite) theory whose models are dense linear orderings with no greatest or least point.

Note that $\mathbb{Q}$ with the usual ordering is a countable DLOWE.

**Theorem 4.** *Any two countable DLOWE's are isomorphic.*

*Proof.* Let $X$ and $Y$ be such orderings. We build an isomorphism by the "back and forth" method. Enumerate the sets $X$ and $Y$ without repetitions as $x_0, x_1, \ldots$ and $y_0, y_1, \ldots$.

We will build a sequence of functions $g_1, g_2 \ldots$ such that $g_i$ is an order-preserving bijection between a set of $i$ elements in $X$ and a set of $i$ elements in $Y$. We will make sure that $g_{i+1} \upharpoonright \mathrm{dom}(g_i) = g_i$. Start by defining $g_1(x_0) = y_0$.

If we defined $g_i$ then:

- If $i$ is odd, find the least $m$ such that $x_m \notin \mathrm{dom}(g_i)$, then the least $n$ such that $y_n \notin \mathrm{rge}(g_i)$, and setting $\mathrm{dom}(g_{i+1}) = \mathrm{dom}(g_i) \cup \{x_m\}$ while defining $g_{i+1}(x_m) = y_n$ gives an order-preserving map. This is possible because $Y$ is a DLOWE. This defines $g_{i+1}$.
- If $i$ is even, find the least $n$ such that $y_n \notin \mathrm{rge}(g_i)$, then the least $m$ such that $x_m \notin \mathrm{dom}(g_i)$ and setting $\mathrm{dom}(g_{i+1}) = \mathrm{dom}(g_i) \cup \{x_m\}$ while defining $g_{i+1}(x_m) = y_n$ gives an order-preserving map. This is possible because $X$ is a DLOWE. This defines $g_{i+1}$.

Taking the union of the $g_i$'s gives a function $g : X \to Y$. It is clearly an order-preserving bijection between some subset of $X$ and some subset of $Y$. To finish we show that $\mathrm{dom}(g) = X$ and $\mathrm{rge}(g) = Y$.

If $\mathrm{dom}(g) \neq X$ then let $m$ be least such that $x_m \notin \mathrm{dom}(g)$. Since $m$ is least, find $i$ which is odd and so large that $\{x_r : r < m\} \subseteq \mathrm{dom}(g_i)$. At stage $i$ we would add $x_m$ to the domain, contradiction. Similarly $\mathrm{rge}(g) = Y$.

$\square$

Since all countable DLOWE's are isomorphic to $\mathbb{Q}$, we can focus on $\mathbb{Q}$.

For any structure $\mathcal{M}$, an *automorphism of $\mathcal{M}$* is an isomorphism from $\mathcal{M}$ to $\mathcal{M}$. Think of this as a symmetry of $\mathcal{M}$.

**Theorem 5.** *If $a_i$ and $b_i$ are rationals with $a_1 < \ldots < a_n$ and $b_1 < \ldots < b_n$, then there is an automorphism $g$ of $\mathbb{Q}$ such that $g(a_i) = b_i$ for all $i$.*

*Proof.* Build the automorphism in stages as in the last proof, but start with $\mathrm{dom}(g_0) = \{a_1, \ldots a_n\}$ and $g_0(a_i) = b_i$. $\square$