SKETCHY NOTES FOR WEEKS 9 AND 10, PART ONE

1. Completeness with equality

So far we did completeness for first order logic without equality. Now we do it for FOL with equality. Many details are the same as before so we focus on the new ideas. Keep in mind that in the setting with FOL with equality the notion of *structure* is different (we must interpret $\equiv$ as equality) and the notion of *proof* is different (we have the equality rules).

Starting with a consistent theory T, we can expand it as before to a Henkinised theory in a larger language. So our task is to construct, given a Henkinised T, a structure $\mathcal{M}$ such that $\mathcal{M} \models T$.

We use our previous version of completeness to construct $\mathcal{N}$ which is a structure (in the old sense) for the language of T, such that $\mathcal{N} \models \phi$ if and only if $\phi \in T$. Recall that the underlying set N for $\mathcal{N}$ is the set of closed terms. The snag is that $\mathcal{N}$ may not interpret $\equiv$ as equality.

We define a relation $\sim$ on N as follows: $\sigma \sim \tau$ if and only if $\sigma \equiv \tau \in T$. The rules for equality tell us that $\sim$ is an equivalence relation on N, for example $\{\sigma \equiv \tau, \tau \equiv \rho\} \vdash \sigma \equiv \rho$ and this is exactly what we need to prove that $\sim$ is transitive.

Let M be the set of equivalence classes, and write $[\sigma]$ for the class of σ. We will define a structure $\mathcal{M}$ with underlying set M, but we have to be slightly careful.

Constant symbols are easy, just define $c^{\mathcal{M}} = [c^{\mathcal{N}}]$. For an n-ary relation symbol R we would like to define that $R^{\mathcal{M}}([\tau_1], \ldots [\tau_n])$ if and only if $R^{\mathcal{N}}(\tau_1, \ldots \tau_n)$, but for this to make sense it is necessary that if $\tau_i \sim \tau_i'$ for each i then $R^{\mathcal{M}}(\tau_1, \ldots \tau_n) \iff R^{\mathcal{M}}(\tau_1', \ldots \tau_n')$. This is guaranteed by one of the proof rules for equality. In a similar vein we define $f^{\mathcal{M}}([\tau_1], \ldots [\tau_n]) = [f(\tau_1, \ldots \tau_n)]$.

Remark: by the definition of $\sim$, $\equiv$ now interprets as equality, so that $\mathcal{M}$ is a structure (in the new sense) for the language of T.

We now verify by induction that for every sentence ϕ, $\phi \in T$ if and only if $\mathcal{M} \models \phi$. First we show by a routine induction that for every closed term τ, $\tau^{\mathcal{M}} = [\tau]$. Then we verify the main claim for atomic sentences ϕ, that is sentences of the form $R(\tau_1, \ldots \tau_n)$: this is easy, $R(\tau_1, \ldots \tau_n) \in T$ if and only if $R^{\mathcal{N}}(\tau_1, \ldots \tau_n)$ (by construction of $\mathcal{N}$) if and only if $R^{\mathcal{M}}([\tau_1], \ldots [\tau_n])$ (by construction of $\mathcal{M}$) if and only if $\mathcal{M} \models R(\tau_1, \ldots \tau_n)$ (since $\tau_i^{\mathcal{M}} = [\tau_i]$).

The rest of the proof is exactly as before.

CONVENTION: FROM THIS POINT ON WE WORK ONLY WITH FOL WITH EQUALITY, UNLESS OTHERWISE SPECIFIED.

2. Reducts and expansions

Definition 1. *Let $\mathcal{L}_0$ and $\mathcal{L}_1$ be languages with $\mathcal{L}_0 \subseteq \mathcal{L}_1$. Let $\mathcal{M}$ be a structure for the language $\mathcal{L}_1$. Then the* reduct *of $\mathcal{M}$ to the language $\mathcal{L}_0$ is the structure for $\mathcal{L}_0$ obtained as follows: the underlying set of the reduct is the underlying set M of* *the structure $\mathcal{M}$, and every symbol of $\mathcal{L}_0$ has the same interpretation in the reduct as it has in the structure $\mathcal{M}$.*

Intuitively we build a reduct by "forgetting" the interpretations of certain symbols. If $\mathcal{M}$ is a reduct of some structure $\mathcal{M}^+$ we say that $\mathcal{M}^+$ is an *expansion* of $\mathcal{M}$.

The following result is essentially obvious. The proof is by induction on the formulae of $\mathcal{L}_0$.

Theorem 1. *Let Γ be a set of sentences of $\mathcal{L}_1$ and let $\mathcal{M}$ be a structure for $\mathcal{L}_1$. Let $\mathcal{M}^*$ be the reduct of $\mathcal{M}$ to $\mathcal{L}_0$. If $\mathcal{M} \models \Gamma$ then $\mathcal{M}^* \models \Gamma^*$ where Γ^* is the intersection of Γ with the sentences of $\mathcal{L}_0$.*

The next result is even easier.

Theorem 2. *If $\mathcal{L}_0 \subseteq \mathcal{L}_1$ and $\mathcal{M}$ is a structure for $\mathcal{L}_0$, then there is a structure $\mathcal{N}$ for $\mathcal{L}_1$ whose reduct to $\mathcal{L}_0$ is $\mathcal{M}$.*

Formally speaking when we proved completness we were actually using these ideas. We started with a consistent theory T, found a Henkinised theory $T' \supseteq T$ in a larger language and built a structure for that larger language which is a model of T'. So we should have taken a reduct to get a structure for the language of T which is a model of T.

3. Easy consequences of completeness

The next result shows that we can stop being careful about which language we are working in, using Completeness.

Theorem 3. *Let Γ be a set of sentences in $\mathcal{L}$, let ϕ be a sentence in $\mathcal{L}$. Suppose that Γ proves ϕ in the proof system for $\mathcal{L}'$ where $\mathcal{L} \subseteq \mathcal{L}'$. Then Γ proves ϕ in the proof system for $\mathcal{L}$.*

Proof. We claim that Γ entails ϕ. To see this let $\mathcal{M}$ be an arbitrary structure for $\mathcal{L}$ which models Γ, and let $\mathcal{M}'$ be any structure for $\mathcal{L}'$ whose reduct to $\mathcal{L}$ is $\mathcal{M}$. By Soundness $\mathcal{M}' \models \phi$, and since ϕ is a sentence in $\mathcal{L}$ we also have $\mathcal{M} \models \phi$.

Since Γ entails ϕ, it follows from Completeness that there is a proof of ϕ from Γ in the $\mathcal{L}$ proof system. $\square$

In particular the notion of "consistency" is rather robust, in a sense made precise by the following easy corollary of the preceding one.

Theorem 4. *Let Γ be a set of sentences in $\mathcal{L}$, and let $\mathcal{L} \subseteq \mathcal{L}'$. Γ is consistent in the proof system for $\mathcal{L}$ if and only if Γ is consistent in the proof system for $\mathcal{L}'$.*

The following fact has some strange consequences (the "Skolem paradox") which we may have time to explore later.

Theorem 5. *Let Γ be a consistent set of sentences in a countable first order language $\mathcal{L}_V$. Then Γ has a countable model (that is to say a model which has a countable underlying set).*

Proof. We can expand to a Henkinised theory in a countable language and then build a model out of closed terms. There are onlycountably many such terms. $\square$

The Compactness theorem is proved just as in the case of propositional logic. As we see later, the fact that the expressive power of first order logic is greater than that of propositional logic makes the Compactness theorem more interesting in the case of first order logic.

Theorem 6 (Compactness)**.** *If Γ entails ϕ then some finite subset of Γ entails ϕ.*

Proof. By Completeness and the fact that proofs are finite. □

4. Elementary embeddings, elementary equivalence

The next definition is a generalisation of an important idea in algebra.

Definition 2. *Let $\mathcal{M}$ and $\mathcal{N}$ be two structures for the same language $\mathcal{L}$. A function f is an* isomorphism from M to N *($f : M \simeq N$) if and only if*

1. *f is a bijection from M to N.*
2. *$f(c^{\mathcal{M}}) = c^{\mathcal{N}}$ for every constant symbol c of V.*
3. *$f(g^{\mathcal{M}}(a_1, \dots a_k)) = g^{\mathcal{N}}(f(a_1), \dots f(a_k))$ for every function symbol g of V and $a_1, \dots a_k \in M$.*
4. *$R^{\mathcal{M}}(a_1, \dots a_k) \iff R^{\mathcal{N}}(f(a_1), \dots f(a_k))$ for every relation symbol R of V and $a_1, \dots a_k \in M$.*

We say that $\mathcal{M}$ and $\mathcal{N}$ are *isomorphic* and write $\mathcal{M} \simeq \mathcal{N}$ when there exists an isomorphism from $\mathcal{M}$ to $\mathcal{N}$.

The following result is easy.

Theorem 7. *Let $\phi(x_1, \dots x_k)$ be a formula of $\mathcal{L}$, and let f be an isomorphism from $\mathcal{M}$ to $\mathcal{N}$. Then for any $a_1, \dots a_k \in M$,*

$$\mathcal{M} \models \phi(a_1, \dots a_k) \iff \mathcal{N} \models \phi(f(a_1), \dots f(a_k)).$$

Corollary 1. *If $\mathcal{M}$ and $\mathcal{N}$ are isomorphic then for every sentence ϕ, $\mathcal{M} \models \phi \iff \mathcal{N} \models \phi$.*

Definition 3. *Two structures $\mathcal{M}$ and $\mathcal{N}$ are* elementarily equivalent (in the language $\mathcal{L}$) *if and only if for every sentence ϕ of $\mathcal{L}$, $\mathcal{M} \models \phi \iff \mathcal{N} \models \phi$.*

We write $\mathcal{M} \equiv \mathcal{N}$ for the relation of elementary equivalence.

We have seen that if $\mathcal{M} \simeq \mathcal{N}$, then $\mathcal{M} \equiv \mathcal{N}$. The converse is false in general (see an example in the next section).

We can use compactness to get interesting examples of elementary embeddings. To do this we associate various theories to an $\mathcal{L}$-structure $\mathcal{M}$.

1. The *theory* of $\mathcal{M}$ is the set of sentences ϕ of $\mathcal{L}$ such that $\mathcal{M} \models \phi$.
2. The *complete diagram* of $\mathcal{M}$ is the set of sentences ϕ of the expanded language for $\mathcal{M}$ such that $\mathcal{M} \models \phi$.
3. The *atomic diagram* of $\mathcal{M}$ is the set of atomic sentences ϕ of the expanded language for $\mathcal{M}$ such that $\mathcal{M} \models \phi$.

Theorem 8. *If $\mathcal{M}$ is an infinite structure for $\mathcal{L}$ then there is an elementary embedding f from $\mathcal{M}$ into some $\mathcal{N}$ where $N \neq \mathrm{rge}(f)$.*

Proof. Let c be a new constant and let T^* be the union of the complete diagram of $\mathcal{M}$ and the set of sentences $\neg c \equiv c_m$ for $m \in M$. Since M is infinite, T^* is consistent. Let $\mathcal{N}$ be a model of T^* and define $f(m) = c_m^{\mathcal{N}}$ for all m. □

5. A "non standard" integer

Let $\mathcal{L}$ be a language with constant symbols 0 and 1, binary function symbols $+$ and $\times$, and binary relation symbols $\equiv$ and $\leq$. We consider $\mathbb{N}$ as a structure for $\mathcal{L}$ in the natural way

Let TA (True Arithmetic) be the complete diagram of $\mathbb{N}$. Let c be a constant symbol not used in V and let

$$T^* = TA \cup \{c > 1, c > 1 + 1, c > 1 + 1 + 1, \ldots\}$$

T^* is consistent. Since T^* is a theory in a *countable* language, it follows by our remarks after the proof of Completeness that T^* has a *countable* model. Let $\mathbb{N}^*$ be a model of T^*. Arguing as in the last section the map $f : n \mapsto c_n^{\mathbb{N}^*}$ is an elementary embedding from $\mathbb{N}$ to $\mathbb{N}^*$.

We claim that $\mathbb{N}$ is not isomorphic to $\mathbb{N}^*$. To see this let g be an isomorphism and observe that by an easy induction $g(n) = f(n)$ for all n, but the element $c^{\mathbb{N}^*}$ is not in the range of f.