SKETCHY NOTES FOR WEEKS 7 AND 8

We are now ready to start work on the proof of the Completeness Theorem for first order logic. Before we start a couple of remarks are in order

(1) When we studied propositional logic we assumed that we started with a countable set $\{A_n : n \in \mathbb{N}\}$ of propositional letters, which implied that the set of all propositional formulae was countable. Countability was not used in most of our development of propositional logic, in fact we used it just once, when we proved Completeness: it was used there to list all the formulae of propositional logic as $\{\phi_n : n \in \mathbb{N}\}$ so that we could deal with them in the course of a certain recursive construction. It can be shown (using a bit of set theory) that the whole theory of propositional logic (including Completeness) goes through if we start off with a set $\{A_i : i \in I\}$ of propositional letters indexed by some uncountable set I.

    When we defined first order logic we allowed the sets of constant, relation and function symbols to be arbitrarily large. This assumption did not cause us any problem so far, but again when we prove Completeness we will have to assume that the set of symbols which appear in the language is countable. This restriction can be removed using ideas from set theory just as in the propositional case.

(2) As we remarked when we defined the notion of proof, the definition of proof is done relative to a fixed first order language $\mathcal{L}$. For example the $\forall$-elimination rule lets us draw the conclusion $\phi[x/t]$ from the hypothesis $\forall x\ \phi$, and what use we can make of this rule depends on what terms t are available.

    This has not been problematic so far, but in the proof of Completeness a key idea will be that we expand a language $\mathcal{L}$ to a larger language $\mathcal{L}'$ which has more constant symbols: this means that the proof system for $\mathcal{L}'$ can build more proofs than the proof system for $\mathcal{L}$. This raises a very ugly possibility: if T is a theory (set of sentences) in $\mathcal{L}$ then maybe T looks consistent in the $\mathcal{L}$ proof system but becomes inconsistent in the larger $\mathcal{L}'$ proof system?

    Actually this ugly possibility can not occur, but we will need the Completeness theorem to see this. So in the course of the proof of Completeness we will be extremely punctilious when talking about proofs and consistency, and will always specify which proof system we mean.

(3) The equality symbol presents some extra difficulties in the proof of Completeness. We will therefore prove two versions of Completeness. First we do one without equality, then we build on that to do a proof for logic with equality.

The following definition is the key one.

Definition 1. *Let T be a set of sentences in a first order language $\mathcal{L}$. T is* Henkinised for $\mathcal{L}$ *if and only if*

(1) *T is consistent in the proof system for $\mathcal{L}$.*

(2) *For every sentence ϕ of $\mathcal{L}$, either ϕ or $\neg\phi$ is in T.*
(3) *For any formula ϕ of $\mathcal{L}$, if $\exists x\ \phi \in T$ then there is a constant symbol c of $\mathcal{L}$ such that $\phi[x/c] \in T$.*

The role of the constants in part 3 of this definition is to provide concrete witnesses to the truth of existential statements. They are often called "Henkin constants" or "Henkin witnesses", after the logician Leon Henkin. On a historical note the first proof of Completeness was given by Gödel using very different ideas: the modern style of proof which we give is due to Henkin.

The rough outline of the proof of Completeness is this:

- Prove that consistent theories can be extended to Henkinised theories (in a larger language).
- Prove that Henkinised theories have models (a model of T is a structure $\mathcal{M}$ such that $\mathcal{M} \models T$)
- Infer that consistent theories have models.
- Now argue as for propositional logic: If $T \not\vdash \phi$ then $T \cup \{\neg\phi\}$ is consistent, so it has a model, so $\mathcal{M} \not\models \phi$.

As an aid to building Henkinised sets we need some lemmas, of which most are just trivial generalisations of results we have seen before.

Lemma 1. *Let T be a set of sentences in $\mathcal{L}$ and let ϕ be a sentence in $\mathcal{L}$. If T does not prove ϕ in the proof system for $\mathcal{L}$, then $T \cup \{\neg\phi\}$ is consistent in the proof system for $\mathcal{L}$.*

Lemma 2. *Let T be a set of sentences in $\mathcal{L}$ and let ϕ be a sentence in $\mathcal{L}_V$. If T does not prove $\neg\phi$ in the proof system for $\mathcal{L}$, then $T \cup \{\phi\}$ is consistent in the proof system for $\mathcal{L}$.*

Lemma 3. *If T is Henkinised, it is deductively closed.*

The following lemma is a bit harder.

Lemma 4. *Let T be a set of sentences in $\mathcal{L}$ which is consistent in the proof system for $\mathcal{L}$. Let c be a constant symbol not appearing in $\mathcal{L}$ and let $\mathcal{L}'$ be the language obtained by adding c to the symbols of $\mathcal{L}$. If $\exists x\ \phi \in T$ then $T \cup \{\phi[x/c]\}$ is consistent in the proof system for $\mathcal{L}'$.*

Proof. Suppose for contradiction that $T \cup \{\phi[x/c]\}$ is inconsistent. Using $\neg$-Introduction we can build a proof P in the $\mathcal{L}'$ system with hypotheses in T and conclusion $\neg\phi[x/c]$. Let y be a variable symbol not equal to x and not appearing in P.

An easy induction shows that if P^* is the tree obtained by replacing each appearance of c in P by y, then P^* is a proof in the $\mathcal{L}$ system. Subtle point: The result of replacing c by y in $\neg\phi[x/c]$ is $\neg\phi[x/y]$, so the conclusion of P^* is $\neg\phi[x/y]$.

Since y has no appearance in the hypotheses of P, we can do $\forall$-Introduction and make a proof of $\forall y\ \neg\phi[x/y]$ in the $\mathcal{L}$ system with hypotheses in T. It is now easy to see that T is inconsistent in the $\mathcal{L}$ system, contradiction! $\square$

Now we prove the main technical result needed for Completeness. The proof is not too long but has one or two subtle aspects (see remarks after the proof). By a "countable langauge" we mean one which has a countable signature, which implies there are countably many formulae.

Theorem 1. *Let $\mathcal{L}$ be a countable language. Let T be a set of sentences in $\mathcal{L}$ which is consistent in the $\mathcal{L}$ proof system. Then there are a countable language $\mathcal{L}'$ extending $\mathcal{L}$ and a set of sentences T' in $\mathcal{L}'$ such that $T \subseteq T'$ and $\bar{\Gamma}$ is Henkinised for $\mathcal{L}'$.*

Proof. We start by fixing a set $\{d_n : n \in \mathbb{N}\}$ of constant symbols which are distinct and do not appear in $\mathcal{L}$. $\mathcal{L}'$ is defined to be the language obtained from $\mathcal{L}$ by adding the set $\{d_n : n \in \mathbb{N}\}$ of constant symbols. We also define $\mathcal{L}_j$ to be the language obtained from $\mathcal{L}$ by adding in the finite set $\{d_i : i < j\}$ and note that $\mathcal{L}_0 = \mathcal{L}$.

The set of formulae of $\mathcal{L}'$ is countable, in particular the set of sentences of $\mathcal{L}'$ is countable. We enumerate this set as $\{\phi_j : j \in \mathbb{N}\}$, in such a way that ϕ_j is a formula of $\mathcal{L}_j$ for every j (THIS IS A KEY TECHNICAL POINT)

We will construct an increasing sequence

$$T_0 \subseteq T_1 \dots$$

such that

1. $T_0 = T$.
2. T_j is a set of sentences in $\mathcal{L}_j$ and is consistent for the $\mathcal{L}_j$ proof system.
3. For all j
   - (a) Either $\phi_j \in T_{j+1}$ or $\neg\phi_j \in T_{j+1}$.
   - (b) If $\phi_j \in T_{j+1}$ and has the form $\exists x\ \psi$ for some ψ, then $\psi[x/d_j] \in T_{j+1}$.

The construction is inductive, starting by setting $T_0 = T$. Suppose that we have constructed T_j. We first define an intermediate set S_j.

If T_j proves ϕ_j in the $\mathcal{L}_j$ proof system then we set $S_j = T_j \cup \{\phi_j\}$, otherwise we set $S_j = T_j \cup \{\neg\phi_j\}$. As we have seen, S_j is consistent in the $\mathcal{L}_j$ proof system.

Now if $\phi_j \in S_j$ and ϕ_j has the form $\exists x\ \psi$ then we set $T_{j+1} = S_j \cup \{\psi[x/d_j]\}$, otherwise we set $T_{j+1} = S_j$. Since the symbol d_j does not appear in the vocabulary V_j, it follows from previous results that T_{j+1} is consistent in the proof system for $\mathcal{L}_{j+1}$.

Now we let $T' = \bigcup_n T_n$ and check that T' is Henkinised for $\mathcal{L}'$.

1. We start by checking that T' is consistent in the $\mathcal{L}'$ proof system. Suppose for a contradiction that this is not so and fix proofs P and Q in the $\mathcal{L}'$ proof system which have conclusions β and $\neg\beta$ for some β, and hypotheses lying in T'.

   Find n so large that all the hypotheses of P and Q appear in T_n, and that all the formulae and cancelled formulae appearing in P and Q come from $\mathcal{L}_n$. Then P and Q are proofs in the proof system for $\mathcal{L}_n$ giving contradictory conclusions, which is impossible since the construction guarantees that T_n is consistent in the $\mathcal{L}_n$ proof system.
2. Next we check that for every sentence ϕ of $\mathcal{L}'$, either ϕ or $\neg\phi$ is in T'. Fix such a ϕ, and let $\phi = \phi_j$. By construction one of ϕ_j and $\neg\phi_j$ lies in T_{j+1}, and since $T_{j+1} \subseteq T'$ we are done.
3. Finally we check that if $\exists x\ \psi \in T'$ then $\psi[x/c] \in T'$ for some c. Let $\exists x\ \psi = \phi_j$ and assume that $\phi_j \in T'$. By construction we know that one of the formulae ϕ_j and $\neg\phi_j$ is in S_j, and since T' is consistent it must be that $\phi_j \in S_j$. By construction $\psi[x/d_j] \in T_{j+1}$ and we are done.

□

Note that

(1) We had to "catch our tails" in the sense that we had to construct a set T' of sentences in the $\mathcal{L}'$ which was Henkinised *for* $\mathcal{L}'$. This is why it was important to enumerate all sentences of $\mathcal{L}'$ before starting to build T'.
(2) It was key that ϕ_j only contained constant symbols d_i for $i < j$. This helped guarantee that S_j only contains d_i for $i < j$, which made it possible to use d_j as the "new constant" serving as a Henkin witness for ϕ_j (in the case when ϕ_j had the form $\exists x\ \psi$).
(3) It was also helpful that when ϕ_j had the form $\exists x\ \psi$ we made sure to deal with it at stage j in the construction. It would have been a nuisance if we had to worry about handling ϕ_j at some later stage.

We will now prove that every Henkinised set of sentences is satisfied by some structure: this is the remaining hard step in the proof of Completeness. Before doing this we review the definition of the satisfaction relation $\mathcal{M} \models \phi$. It will be helpful to view the definition of satisfaction as happening in two phases, each of them inductive:

- In phase 1 we define $\mathcal{M} \models \phi$ for sentences ϕ which contain no quantifiers. The base case here is ϕ having the form $R(t_1, \dots t_k)$ for a relation symbol R and closed terms t_i; the successor steps are given by the clauses for the four propositional connectives. If we like to, we can view this definition as proceeding by induction on the number of connectives appearing in ϕ.
- In phase 2 we define $\mathcal{M} \models \phi$ for arbitrary sentences ϕ, by induction on the number of quantifiers appearing in ϕ. The base case here is ϕ without any quantifiers, where we use the definition from phase 1. After n steps in the phase 2 construction we have defined $\mathcal{M} \models \phi$ for ϕ having at most n quantifiers.

  The definition of $\mathcal{M} \models \phi$ for ϕ with $n+1$ quantifiers again proceeds by induction on the number of connectives appearing in ϕ. If ϕ has any of the forms $\neg\psi$, $\psi \wedge \chi$, $\psi \vee \chi$, $\psi \to \chi$ then each of ψ and χ has at most $n+1$ quantifiers and has fewer connectives than ϕ; so we have already defined $\mathcal{M} \models \psi$ and $\mathcal{M} \models \chi$ and can proceed.

  If ϕ has the form $Qx\ \psi$ where Q is a quantifier then all the formulae of the form $\psi[x/c_a]$ for $a \in M$ have exactly n quantifiers, so again we have already defined $\mathcal{M} \models \psi[x/c_a]$ and can proceed.

We are now ready to build a model of a Henkinised set of sentences.

Theorem 2. *Let T be a set of sentences in $\mathcal{L}$, where $\mathcal{L}$ has at least one constant symbol. Let T be Henkinised for $\mathcal{L}$. Then there exists a structure $\mathcal{M}$ for $\mathcal{L}$ such that $\mathcal{M} \models \Gamma$.*

Proof. We let M be the set of all closed terms for the language $\mathcal{L}$. Since there is at least one constant symbol, M is not empty. We will build a structure $\mathcal{M}$ with underlying set M; note that this is potentially rather confusing since it blurs the distinction between syntax and semantics.

To define $\mathcal{M}$ we need to describe how the constant, function and relation symbols of V are to be interpreted in $\mathcal{M}$.

- $c^{\mathcal{M}} = c$ for every constant synbol c.
- If f is a function symbol of arity k, then $f^{\mathcal{M}}(t_1, \dots t_k) = f(t_1, \dots t_k)$ for all $t_1, \dots t_k \in M$. NOTE: on the left hand side of this equation we have

the function $f^{\mathcal{M}}$ applied to the objects $t_1, \dots t_k$ of the structure M, on the right we have the closed term $f(t_1, \dots t_k)$.

- If R is a relation symbol of arity k and $t_1, \dots t_k \in M$ then $R^{\mathcal{M}}(t_1, \dots t_k)$ if and only if $R(t_1, \dots t_k) \in T$. NOTE: this is actually very reasonable, since we are trying to cook up a structure which believes all the sentences in T.

Lemma 5. *For every closed term t of $\mathcal{L}_V$, $t^{\mathcal{M}} = t$.*

Proof. The proof is an easy induction on the construction of closed terms. By definition $c^{\mathcal{M}} = c$ for all c, and for the induction step we have that

$$(f(t_1, \dots t_k))^{\mathcal{M}} = f^{\mathcal{M}}(t_1^{\mathcal{M}}, \dots t_k^{\mathcal{M}}) = f^{\mathcal{M}}(t_1, \dots t_k) = f(t_1, \dots t_k)$$

where the first equality is by the definition of the interpretation of a closed term, the second equality is by induction, and the third equality is by the definition of $f^{\mathcal{M}}$. $\square$

We will now show that for every sentence ϕ in $\mathcal{L}$, $\phi \in T$ if and only if $\mathcal{M} \models \phi$. This will be shown by induction on ϕ. KEY POINT: I will organise the inductive proof in exactly the same way as I recently organised the inductive definition of satisfaction.

- Phase 1: we prove that for every sentence ϕ with no quantifiers, $\phi \in T$ if and only if $\mathcal{M} \models \phi$. We start with the atomic sentences and then proceed by induction on the number of connectives in ϕ.
  - Base case: $\phi = R(t_1, \dots t_p)$ where R is a relation symbol and the t_i are closed terms.
    By the definition of satisfaction, $\mathcal{M} \models \phi$ if and only if $R^{\mathcal{M}}(t_1^{\mathcal{M}}, \dots t_k^{\mathcal{M}})$. By the lemma we just proved on interpretation of closed terms in $\mathcal{M}$, $t_i^{\mathcal{M}} = t_i$ for all i. By the definition of $R^{\mathcal{M}}$, $R^{\mathcal{M}}(t_1, \dots t_k)$ if and only if $R(t_1, \dots t_k) \in T$.
    It follows that $\phi \in T$ if and only if $\mathcal{M} \models \phi$, and we have established the base case of the induction.
  - Successor steps: there is one successor step for each of the four propositional connectives. These are very similar to the corresponding steps in the proof of Completeness for propositional logic, so we only do the step for negation.
    Let $\phi = \neg\psi$. By the definition of satisfaction, $\mathcal{M} \models \phi$ if and only if $\mathcal{M} \not\models \psi$. By induction $\mathcal{M} \not\models \psi$ if and only if $\psi \notin T$. Since T is Henkinised, $\psi \notin T$ if and only if $\neg\psi \in T$.

  At the end of phase 1 we have established that if ϕ is a sentence with no quantifiers then $\phi \in T$ if and only if $\mathcal{M} \models \phi$.
- Phase 2: we prove that for every sentence ϕ, $\phi \in T$ if and only if $\mathcal{M} \models \phi$. We proceed by induction on the number of quantifiers in ϕ; the case where ϕ has no quantifiers has already been handled by phase 1 of the argument.

  For the induction step, suppose that we have shown that for every sentence ϕ with at most n quantifiers, $\phi \in T$ if and only if $\mathcal{M} \models \phi$. We will now establish the same statement for ϕ with $n+1$ quantifiers, by induction on the number of connectives appearing in ϕ.

  There are various cases, depending on how the formula ϕ was constructed. The steps for the propositional connectives are again similar to those for the proof of Completeness in propositional logic, so again we will only do one step in detail: for variety we do the step for the $\wedge$ connective.

- Let $\phi = \psi \wedge \chi$. The formulae ψ and χ have at most $n+1$ quantifiers and have fewer connectives than does ϕ, so by induction we know that $\mathcal{M} \models \psi$ if and only if $\psi \in \Gamma$, and similarly for χ.
  By the definition of satisfaction $\mathcal{M} \models \phi$ if and only if $\mathcal{M} \models \psi$ and $\mathcal{M} \models \chi$, which in turn is true if and only if $\psi \in T$ and $\chi \in T$.
  Since T is Henkinised it is deductively closed, it is easy to check that $\psi \in \Gamma$ and $\chi \in \Gamma$ if and only if $\phi \in \Gamma$. The point is that $\{\psi, \chi\} \vdash \phi$ and $\phi \vdash \{\psi, \chi\}$.

Now we deal with the quantifiers. The case for the existential quantifier represents the key point in the argument; it is here that we use the "Henkin witnesses".

- Let $\phi = \exists x\ \psi$. Suppose firstly that $\mathcal{M} \models \phi$. By definition there is an element t of M such that $\mathcal{M} \models \psi[x/c_t]$. Now $c_t^{\mathcal{M}} = t = t^{\mathcal{M}}$, and so $\mathcal{M} \models \psi[x/t]$.
  The sentence $\psi[x/t]$ is a sentence of $\mathcal{L}$ with n quantifiers, and so by induction $\psi[x/t] \in T$. Since $\psi[x/t] \vdash \phi$ and T is deductively closed, it follows that $\phi \in \Gamma$.
  Conversely suppose that $\phi \in T$. Since T is Henkinised, there is some constant symbol d such that $\psi[x/d] \in T$. By induction $\mathcal{M} \models \psi[x/d]$, and so $\mathcal{M} \models \exists x\ \psi$.
- Let $\phi = \forall x\ \psi$. Suppose firstly that $\mathcal{M} \models \forall x\ \psi$, and suppose for a contradiction that $\forall x\ \psi \notin T$. Since T is Henkinised $\neg\forall x\ \psi \in T$, and therefore $\exists x\ \neg\psi \in T$.
  Since T is Henkinised there is a constant symbol d such that $\neg\psi[x/d] \in T$. By induction $\mathcal{M} \models \neg\psi[x/d]$, contradicting our assumption that $\mathcal{M} \models \forall x\ \psi$.
  Now suppose $\mathcal{M} \not\models \forall x\ \psi$, so that by definition $\mathcal{M} \models \neg\psi[x/c_a]$ for some $a \in M$. Since $a^{\mathcal{M}} = a = c_a^{\mathcal{M}}$, $\mathcal{M} \models \neg\psi[x/a]$. By induction $\neg\psi[x/c_a] \in T$, and so since T is consistent $\forall x\ \psi \notin T$.

This concludes the inductive proof that for every sentence ϕ of $\mathcal{L}$, $\phi \in T$ if and only if $\mathcal{M} \models \phi$. It follows trivially that $\mathcal{M} \models T$.

$\square$