SKETCHY NOTES FOR WEEK 5

We will start with the proof rules for propositional logic and add Intro and Elim rules for the quantifiers.

Subtle point: In proofs we reason with formulae, not just sentences. This is actually quite natural: in everyday mathematical reasoning you might be given some problem like "prove every even $n > 2$ is composite" and your proof would contain lines where you talked about n without quantifying over it.

It's tempting to think we can always infer $\phi[x/\tau]$ from $\forall x\ \phi$ but this is actually dangerous. Consider the formula $\forall x_0\ \exists x_1\ R(x_0, x_1)$, and compare it with the formula $\exists x_1\ R(x_1, x_1)$.

The problem is that we can create new bindings of variables by substitution. To avoid this we define when it is allowed to substitute τ for x in ϕ, intuitively the idea is that variables appearing in τ should not be bound by quantifiers appearing in ϕ.

- If ϕ is atomic, τ is allowed for x in ϕ.
- τ is allowed for x in $\neg\psi$ if and only if τ is allowed for x in ψ.
- τ is allowed for x in $\phi @ \psi$ if and only if τ is allowed for x in ϕ and τ is allowed for x in ψ.
- τ is allowed for x in $Qy\psi$ if and only if EITHER x has no free appearances in $Qy\psi$, OR τ is allowed for x in ψ and y does not appear in τ.

Note that: x is always allowed for x in ψ.

The rules:

- $\forall$-elimination: If P is a proof with conclusion $\forall x\phi$ and τ is allowed for x in ϕ, then
$$\frac{P}{\phi[x/\tau]}$$
is a proof.

  As we already saw, the restriction is important.
- $\forall$-introduction: If P is a proof with conclusion ϕ and x has no free appearances in the hypotheses of P, then
$$\frac{P}{\forall x\ \phi}$$
is a proof.

  Another cautionary example of a "proof" that one could build by ignoring the restriction:
$$\frac{R(x)}{\forall x\ R(x)}$$
This is clearly silly, being given an example of an x with property R should not let us conclude that everything has property R.
- $\exists$-introduction: If P is a proof with conclusion $\phi[x/\tau]$ where τ is allowed for x in ϕ, then
$$\frac{P}{\exists x\ \phi}$$

is a proof.

- $\exists$-elimination: If P is a proof with conclusion $\exists x\ \phi$, and Q is a proof with conclusion ψ where x has no free appearance in ψ and the only free appaearances of x among the hypotheses of Q are in instances of ϕ, then

$$\frac{P \quad Q^*}{\psi}$$

is a proof where Q^* is obtained from Q by cancelling appearances of ϕ in the hypotheses.

Two more cautionary examples. Make the easy remark that x is always allowed for x so that inferences of the shapes

$$\frac{\forall x \phi}{\phi}$$

and

$$\frac{\phi}{\exists x\ \phi}$$

are always OK.

The following is a valid proof.

$$\dfrac{\dfrac{R(x) \quad Q(x)}{R(x) \wedge Q(x)}}{\exists x\ R(x) \wedge Q(x)}$$

However the following "proof" is clearly silly.

$$\dfrac{\exists x\ Q(x) \quad \dfrac{\exists x\ R(x) \quad \dfrac{\dfrac{\cancel{R(x)} \quad \cancel{Q(x)}}{R(x) \wedge Q(x)}}{\exists x\ R(x) \wedge Q(x)}}{\exists x\ R(x) \wedge Q(x)}}{\exists x\ R(x) \wedge Q(x)}$$

The problem was in the first application of the exists elim rule.

If we are doing first order logic with equality then we need some special rules for the special equality symbol $\equiv$. These rules are:

- =-Refl: for any term σ,

$$\overline{\sigma \equiv \sigma}$$

  is a proof.
- =-Symm: If P is a proof with conclusion $\sigma \equiv \tau$ then

$$\frac{P}{\tau \equiv \sigma}$$

  is a proof.
- =-Trans: If P and Q are proofs with conclusions $\rho \equiv \sigma$ and $\sigma \equiv \tau$ respectively, then

$$\frac{P \quad Q}{\rho \equiv \tau}$$

  is a proof.
- =-Subst-term: If P is a proof with conclusion $x \equiv \tau$ and σ is a term then

$$\frac{P}{\sigma \equiv \sigma[x/\tau]}$$

is a proof.

– =-Subst-form: If P is a proof with conclusion $x = \tau$ for some variable symbol x and term τ, and ϕ is a formula such that τ is allowed for x in ϕ,

$$\frac{P}{\phi \to \phi[x/\tau]}$$

and

$$\frac{P}{\phi[x/\tau] \to \phi}$$

are proofs.

In order to formulate soundness, we need to get rid of the free variables in the hypotheses and conclusion of a proof. Important warning: The free variables of the conclusion may not be the same as the free variables of the hypotheses.

If ϕ is a formula, the *universal closure of* ϕ is the formula obtained by quantifying over the free variables of ϕ. In class I defined it by saying that we list the free variables in increasing order and quantify over them in that order, then I pointed out that it really makes no difference (by results from this week's HW).

Given a proof P, I made a non-standard definition and said that the *sentence of* P is the universal closure of the formula $\phi_1 \wedge \ldots \wedge \phi_t \to \psi$ where the ϕ_i are the hypotheses of P and ψ is the conclusion. In the special case where P has no hypotheses it's just the universal closure of ψ.

Soundness theorem: For any *non-empty* structure $\mathcal{M}$ and any proof P, $\mathcal{M}$ satisfies the sentence of P.

In class I think I forgot the non-empty qualification. One of the rules (which?) becomes unsound if we permit empty structures.

Start of soundness proof (one of the trickier steps): Consider the proof

$$\frac{\forall x\, \phi}{\phi[x/\tau]}$$

where τ is allowed for x in ϕ. The free variables will be $(Free(\phi) \setminus \{x\}) \cup Free(\tau)$.