SKETCHY NOTES FOR WEEK 3 OF BASIC LOGIC, PART THREE

1. Completeness

The proof of the Completeness theorem needs a few preparatory lemmas. Here is a brief sketch of how the proof will eventually go: we will show that if Γ does not prove δ, then there is a truth assignment f such that f satisfies Γ but f does not satisfy δ. That will show that Γ does not entail δ.

Definition 1. *Let Γ be a theory.*

1. Γ *is* consistent *if and only if for every β, Γ proves at most one of β and $(\neg\beta)$.*
2. Γ *is* complete *if and only if for every β, Γ proves at least one of β and $(\neg\beta)$.*
3. Γ *is* deductively closed *if and only if for every β, if $\Gamma \vdash \beta$ then $\beta \in \Gamma$.*

Notice that Γ is contained in its deductive closure.

Lemma 1. *If Γ does not prove δ, then $\Gamma \cup \{(\neg\delta)\}$ is consistent.*

Proof. Suppose for contradiction that $\Gamma \cup \{(\neg\delta)\}$ is not consistent. Fix proofs P_1, P_2 with hypotheses in $\Gamma \cup \{(\neg\delta)\}$ such that P_1 has conclusion β, P_2 has conclusion $(\neg\beta)$.

Using the $\neg$-introduction rule and the contradiction rule we build a proof

$$\dfrac{\dfrac{P_1^* \quad P_2^*}{(\neg(\neg\delta))}\ (\neg I)}{\delta}\ (RAA)$$

where P_i^* is obtained from P_i by cancelling appearances of $(\neg\delta)$ in the hypotheses. The hypotheses of this proof are in Γ, so Γ proves δ. □

Lemma 2. *If Γ does not prove $(\neg\delta)$, then $\Gamma \cup \{\delta\}$ is consistent.*

Proof. Suppose for contradiction that $\Gamma \cup \{\delta\}$ is not consistent. Fix proofs P_1, P_2 with hypotheses in $\Gamma \cup \{\delta\}$ such that P_1 has conclusion β, P_2 has conclusion $(\neg\beta)$.

Using the $\neg$-introduction rule we build a proof

$$\dfrac{P_1^* \quad P_2^*}{\neg\delta}\ (\neg I)$$

where P_i^* is obtained from P_i by cancelling appearances of δ in the hypotheses. The hypotheses of this proof are in Γ, so Γ proves $\neg\delta$. □

To prove the Completeness theorem we will also need the technical concept of *countability.*

Definition 2. *A set X is* countably infinite *if and only if there is a bijection between $\mathbb{N}$ and X. X is* countable *if and only if X is finite or countably infinite.*

Examples of countably infinite sets include $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and the set of all finite subsets of $\mathbb{N}$.

The following fact is standard: if X is countably infinite then so is the set of all finite strings from X. It is a also standard fact that a subset of a countable set is countable.

Lemma 3. *The set of wffs is countably infinite.*

Proof sketch. The set S of symbols used in building formulae of propositional logic is countable. So the set of all finite strings from S is countable, hence the set of wff's is countable. $\square$

The next result contains the main idea in the proof of the Completeness theorem.

Lemma 4. *Let Δ be a consistent theory. Then there exists a theory Δ^* such that*

(1) $\Delta \subseteq \Delta^*$.
(2) Δ^* *is consistent.*
(3) *For all formulae γ, either $\gamma \in \Delta^*$ or $\neg\gamma \in \Delta^*$.*

Proof. Enumerate the wff's as $\langle \gamma_n : n \in \mathbb{N} \rangle$. We will build by recursion a sequence of sets of formulae $\langle \Delta_n : n \in \mathbb{N} \rangle$ with the following properties:

(1) $\Delta_0 = \Delta$.
(2) Δ_n is consistent.
(3) $\Delta_n \subseteq \Delta_{n+1}$.
(4) One of the formulae γ_n, $(\neg\gamma_n)$ is in the set Δ_{n+1}.

Suppose that we have constructed Δ_n. Since Δ_n is consistent, it does not prove both of the formulae γ_n, $(\neg\gamma_n)$.

If Δ_n does prove γ_n then Δ_n does not prove $(\neg\gamma_n)$, and we set $\Delta_{n+1} = \Delta_n \cup \{\gamma_n\}$. If Δ_n does not prove γ_n, then we set $\Delta_{n+1} = \Delta_n \cup \{(\neg\gamma_n)\}$.

We now let $\Delta^* = \bigcup_n \Delta_n$. It is clear that $\Delta \subseteq \Delta^*$, and that Δ^* contains at least one of γ and $(\neg\gamma)$ for every γ.

To see that Δ^* is consistent, suppose for a contradiction that Δ^* is not consistent. Fix proofs P and Q with hypotheses in Δ^*, such that P has conclusion β and Q has conclusion $(\neg\beta)$. Since the proofs P and Q each have finite sets of hypotheses, and the sequence of sets $\langle \Delta_n : n \in \mathbb{N} \rangle$ is increasing, we may fix some integer N sufficiently large that all the hypotheses of P and Q lie in Δ_N. This contradicts the consistency of Δ_N. $\square$

Notice that the Δ^* we have just constructed will contain exactly one of γ and $(\neg\gamma)$ for every γ, since it is consistent.

Lemma 5. *If Δ is consistent and Δ contains at least one of γ and $(\neg\gamma)$ for every γ, then Δ is deductively closed.*

Proof. Let Δ prove ψ. Then it must be that $\psi \in \Delta$, for if not then $(\neg\psi) \in \Delta$ which contradicts the consistency of Δ. $\square$

Before starting the next lemma we make a notational convention: if Γ and Δ are sets of formulae then we say $\Gamma \vdash \Delta$ if and only if $\Gamma \vdash \delta$ for every $\delta \in \Delta$.

Lemma 6. *Let Δ be consistent and suppose that for every ϕ, either $\phi \in \Delta$ or $(\neg\phi) \in \Delta$. There is a truth assignment f such that for all wff's γ, $f \models \gamma \iff \gamma \in \Delta$.*

Proof. For every n, either $A_n \in \Delta$ or $(\neg A_n) \in \Delta$. We define f by setting $f(A_n) = T$ if $A_n \in \Delta$, and $f(A_n) = F$ if $(\neg A_n) \in \Delta$.

Now we show by induction on the inductive structure for propositional logic that for every formula ϕ, $f \models \phi \iff \phi \in T$. Notice that $f \not\models \phi$ if and only if $f \models \neg\phi$.

$\phi = (\neg\psi)$. By the construction of Δ, $\phi \in \Delta$ if and only if $\psi \notin \Delta$. By induction $\psi \notin \Delta$ if and only if $f \models \neg\psi$. Putting these various equivalences together, $\phi \in \Delta$ if and only if $f \models \phi$.

$\phi = (\psi_1 \wedge \psi_2)$.

If $\phi \in \Delta$, then since Δ is deductively closed and $\{\phi\} \vdash \{\psi_1, \psi_2\}$ it follows that $\psi_1 \in \Delta$ and $\psi_2 \in \Delta$. By induction $f \models \psi_1$ and $f \models \psi_2$, hence $f \models \phi$.

If $f \models \phi$ then $f \models \psi_1$ and $f \models \psi_2$, hence by induction ψ_1 and ψ_2 are both in Δ. Since Δ is deductively closed and $\{\psi_1, \psi_2\} \vdash \phi$, it follows that $\phi \in \Delta$.

$\phi = (\psi_1 \vee \psi_2)$.

It turns out to be easier to check that $\phi \notin \Delta$ if and only if $f \models \neg\phi$.

If $\phi \notin \Delta$, then $(\neg\phi) \in \Delta$. Now $(\neg\phi) \vdash \{(\neg\psi_1), (\neg\psi_2)\}$, so $(\neg\psi_1)$ and $(\neg\psi_2)$ are in Δ. Therefore $\psi_1 \notin \Delta$ and $\psi_2 \notin \Delta$, and so by induction $f \models \neg\psi_1$ and $f \models \neg\psi_2$, so $f \models \neg\phi$.

Conversely suppose that $f \models \neg\phi$, so that $f \models \neg\psi_1$ and $f \models \neg\psi_2$. By induction $(\neg\psi_1)$ and $(\neg\psi_2)$ are in Δ, and using the fact that $\{(\neg\psi_1), (\neg\psi_2)\} \vdash (\neg\phi)$ we conclude that $\neg\phi \in \Delta$ and so $\phi \notin \Delta$.

$\phi = (\psi_1 \to \psi_2)$.

As in the preceding case it is easier to show that $\phi \notin \Delta$ if and only if $f \models \neg\phi$.

If $\phi \notin \Delta$ then $(\neg\phi) \in \Delta$. Since $\{(\neg\phi)\} \vdash \{\psi_1, (\neg\psi_2)\}$, $\psi_1 \in \Delta$ and $(\neg\psi_2) \in \Delta$. By induction $f \models \psi_1$ and $f \models \neg\psi_2$, so $f \models \neg\phi$.

Now suppose that $f \models \neg\phi$, so that by definition $f \models \psi_1$ and $f \models \neg\psi_2$. Using the fact that $\{\psi_1, (\neg\psi_2)\} \vdash (\neg\phi)$ we can argue in a familiar way that $\phi \notin \Delta$. □

Exercise 1. *The proof of the last result depended on the following claims. Verify them (they are all pretty easy!)*

- $\{(\psi_1 \wedge \psi_2)\} \vdash \{\psi_1, \psi_2\}$.
- $\{\psi_1, \psi_2\} \vdash (\psi_1 \wedge \psi_2)$.
- $\{(\neg(\psi_1 \vee \psi_2))\} \vdash \{(\neg\psi_1), (\neg\psi_2)\}$.
- $\{(\neg\psi_1), (\neg\psi_2)\} \vdash (\neg(\psi_1 \vee \psi_2))$.
- $\{(\neg(\psi_1 \to \psi_2))\} \vdash \{\psi_1, (\neg\psi_2)\}$.
- $\{\psi_1, (\neg\psi_2)\} \vdash (\neg(\psi_1 \to \psi_2))$.

Now for the Completeness theorem.

Theorem 1 (Completeness for propositional logic)**.** *If* $\Gamma \models \delta$ *then* $\Gamma \vdash \delta$.

Proof. Suppose that $\Gamma \not\vdash \delta$. Then the theory $\Delta = \Gamma \cup \neg\delta$ is consistent. Using the results above we can extend Δ to a consistent Δ^* such that for every γ either γ or $\neg\gamma$ is in Δ^*, and then find f such that $f \models \Delta^*$. So $f \models \Gamma$ and $f \models \neg\delta$, and thus $\Gamma \not\models \delta$. □