SKETCHY NOTES FOR WEEK 3 OF BASIC LOGIC, PART TWO

We now start to build a bridge between the semantic notion of entailment and the syntactic notion of provability.

Definition 1. *Let Γ be a theory and δ be a wff. Then Γ* proves *δ (written $\Gamma \vdash \delta$) if and only if there is a proof P such that the hypotheses of P are all in Γ, and the conclusion of P is δ.*

Our goal for the next couple of lectures is to prove the following pair of theorems.

Soundness theorem: if $\Gamma \vdash \delta$ then $\Gamma \models \delta$.

Completeness theorem: if $\Gamma \models \delta$ then $\Gamma \vdash \delta$.

Notice that we are showing some very different kinds of statement to be equivalent. Truth assignments are infinite objects while proofs are finite. Moreover $\Gamma \models \delta$ is a statement asserting that **for every** truth assignment S some statement about S, Γ, δ holds, while $\Gamma \vdash \delta$ asserts that **there exists** a proof P such that some statement about S, Γ, δ holds.

1. Soundness

It should come as no surprise that the Soundness theorem is proved by induction on proofs. Since there are nine (!) proof rules for generating new proofs from old ones, the proof is quite tedious.

Proof of the Soundness theorem: We prove by induction the following statement about P: every truth assignment f which satisfies the hypotheses of P also satisfies the conclusion of P. For brevity we will say that "P is sound" when it has this property.

Base case: let α be any formula, so that we can treat α as a proof with hypothesis α and conclusion α. It is obvious that α is sound.

Induction step for the $\wedge$-introduction rule.

Let P_1 and P_2 be sound proofs with conclusions α and β respectively, and let R be the proof

$$\frac{P_1 \quad P_2}{(\alpha \wedge \beta)} \ (\wedge I)$$

The set of hypotheses of R is the union of the sets of hypotheses of P_1 and P_2. If f satisfies the hypotheses of R then, since P_1 and P_2 are sound and their hypotheses are satisfied, f satisfies α and f satisfies β. Then by the definition of satisfaction f satisfies $(\alpha \wedge \beta)$.

Induction step for the $\wedge$-elimination rule.

Let P be a sound proof with conclusion $(\alpha \wedge \beta)$, and let R_1, R_2 be the proofs

$$\frac{P}{\alpha} \ (\wedge E)$$

and

$$\frac{P}{\beta}\ (\wedge E)$$

If f satisfies the hypotheses of P, then since P is sound f satisfies $(\alpha \wedge \beta)$. By the definition of satisfaction f satisfies both α and β, that is to say it satisfies the conclusions of R_1 and R_2.

Induction step for the $\vee$-introduction rule.

Let P be a sound proof with conclusion α, let β be any wff, and let R_1, R_2 be the proofs

$$\frac{P}{(\alpha \vee \beta)}\ (\vee I)$$

and

$$\frac{P}{(\beta \vee \alpha)}\ (\vee I)$$

If f satisfies the hypotheses of P, then by soundness of P it satisfies α, hence by the defintion of satisfaction it satisfies the conclusions of R_1 and R_2.

Induction step for the $\vee$-elimination rule.

Let P_1 and P_2 be sound proofs with conclusion γ, and Q be a sound proof with conclusion $(\alpha \vee \beta)$. Let R be the proof

$$\frac{Q \quad P_1^* \quad P_2^*}{\gamma}\ (\vee E)$$

where P_1^* is obtained from P_1 by cancelling occurrences of α in the hypotheses of P_1, and P_2^* is obtained from P_2 by cancelling occurrences of β in the hypotheses of P_2.

The set of hypotheses of R is the union of the hypotheses of Q, the hypotheses of P_1 except for α and the hypotheses of P_2 except for β. Note that α, β may or may not appear as hypotheses for R.

Let f satisfy the hypotheses of R. Then f satisfies the hypotheses of the sound proof Q, so f satisfies $(\alpha \vee \beta)$, so by the definition of satisfaction f satisfies α or S satisfies β.

If f satisfies α then f satisfies all the hypotheses of the sound proof P_1, so f satisfies γ. If F satisfies β then f satisfies all the hypotheses of the sound proof P_2, so again f satisfies γ.

Induction step for the $\to$-introduction rule.

Let P be a sound proof with conclusion β, let α be a wff and let R be the proof

$$\frac{P^*}{\alpha \to \beta}\ (\to I)$$

where P^* is obtained from P by cancelling occurrences of α among the hypotheses of P.

If f satisfies the hypotheses of R then we distinguish two cases. If f satisfies $\neg\alpha$ then it satisfies $\alpha \to \beta$ by definition. If f satisfies α then it satisfies all hypotheses of the sound proof P, hence it satisfies β, and again it satisfies $\alpha \to \beta$ by definition.

Induction step for the $\to$-elimination rule.

Let P_1 be a sound proof with conclusion α, let P_2 be a sound proof with conclusion $(\alpha \to \beta)$, and let R be the proof

$$\frac{P_1 \quad P_2}{\beta} \ (\to E)$$

If f satisfies the hypotheses of R, it satisfies the hypotheses of the sound proofs P_1 and P_2. So it satisfies the conclusions α and $\alpha \to \beta$, so it must satisfy β.

Induction step for the $\neg$-introduction rule.

Let P_1, P_2 be sound proofs with conclusions $\beta, (\neg\beta)$ respectively, let α be a wff and let R be the proof

$$\frac{P_1^* \quad P_2^*}{(\neg\alpha)} \ (\neg I)$$

where P_i^* is obtained from P_i by cancelling occurrences of α among the hypotheses.

Let f satisfy all hypotheses of R. If f were to satisfy α, then it would satisfy all the hypotheses of the sound proofs P_1 and P_2, hence it would satisfy β and $(\neg\beta)$. This is impossible, so f satisfies $\neg\alpha$.

Induction step for the $\neg$-elimination rule.

Let P_1, P_2 be sound proofs with conclusions $\beta, (\neg\beta)$, let γ be a wff and let R be the proof

$$\frac{P_1 \quad P_2}{\gamma} \ (\neg E)$$

If f were to satisfy the hypotheses of R then it would satisfy all hypotheses of the sound proofs P_1, P_2 and hence it would satisfy β and $\neg\beta$. So no such f exists, and vacuously every such f satisfies γ.

Induction step for the Contradiction rule.

Let P be a sound proof with conclusion $(\neg(\neg\beta))$ and let R be the proof

$$\frac{P}{\beta} \ (RAA)$$

If f satisfies the hypotheses of R, then f satisfies $\neg(\neg\beta)$, hence it satisfies β.

$\square$