SKETCHY NOTES FOR WEEK 3 OF BASIC LOGIC, PART ONE

1. Trees

In lecture we took the notion of a *tree of formulae and cancelled formulae* for granted. In these notes we show how to make this notion mathematically precise (you can skip this section if you want, we don't need it later)

We write Seq for the set of all finite sequences of elements of $\mathbb{N}$, and $\lhd$ for the relation "is an initial segment", and $\frown$ for the opertaion of concatenation. The following definitions are not completely standard, but are convenient for us.

Definition 1. *T is a* tree *if and only if*

- $T \subseteq \text{Seq}$.
- *For all $t \in T$ and all $s \lhd t$, $s \in T$.*
- $\langle\rangle \in T$.

Definition 2. *Let T be a tree.*

- $\text{Lev}_n(T) = \{s \in T : \mathit{length}(s) = n\}$.
- *t is an* immediate successor *of s if and only if $\mathit{length}(t) = \mathit{length}(s) + 1$ and $s \lhd t$.*
- *t is a* maximal point *of T if and only if t has no immediate successors in T.*

Definition 3. *A* standard tree *is a tree T such that for all $s \in j$ and $i, j \in \mathbb{N}$, if $s \frown \langle j\rangle \in T$ and $i < j$ then $s \frown \langle i\rangle \in T$.*

For X some nonempty set, a tree of elements of X (X-tree) is a function from some standard tree to X.

Trees lend themselves to construction by recursion. Let $P_0, \ldots P_{n-1}$ be X-trees and let $x \in X$. Then

$$\frac{P_0 \quad \ldots \quad P_{n-1}}{x}$$

is the X-tree Q such that

- $\text{dom}(Q) = \{\langle\rangle\} \cup \{\langle i\rangle \frown s : i < n, s \in \text{dom}(P_i)\}$.
- $Q(\langle\rangle) = x$.
- $Q(\langle i\rangle \frown s) = P_i(s)$.

2. Proofs

A proof is a tree of formulae and *cancelled formulae*. We make the idea of "cancelling a formula" precise, by adding a new symbol / to the language.

Definition 4. *A* cancelled formula *is a string $/\alpha$ where α is a wff.*

We will usually write $\not\alpha$ for this string.

It will follow from the definition that cancelled formulae can only appear at the top of a proof. The *hypotheses* of a proof are the wff's appearing at the top, and the *conclusion* is the wff appearing at the bottom. We will give an inductive structure that generates proofs, which will have a simple "base set" but many functions each corresponding to a "proof rule".

Description of the set B:

For any wff α, the tree whose only entry is α is a proof. We abuse notation by writing "α" for this tree.

Description of the set K:

$\wedge$-introduction rule: If P_1 is a proof with conclusion α, P_2 is a proof with conclusion β then

$$\frac{P_1 \quad P_2}{(\alpha \wedge \beta)}$$

is a proof.

Notice that this rule is implemented by a binary function on the set of trees, which would take as argument the pair (P_1, P_2) and return the tree $\dfrac{P_1 \quad P_2}{(\alpha \wedge \beta)}$

It is sometimes helps the readability of a proof to add a notation saying (in abbreviated form) which proof rule has been used. With this convention we could write the proof above as

$$\frac{P_1 \quad P_2}{(\alpha \wedge \beta)} \ (\wedge I)$$

We have followed this convention with each proof rule given below,.

$\wedge$-elimination: if P is a proof with conclusion $(\alpha \wedge \beta)$, then

$$\frac{P}{\alpha} \ (\wedge E)$$

and

$$\frac{P}{\beta} \ (\wedge E)$$

are proofs.

Notice that that this rule is implemented by a pair of functions, each which has as domain the set of trees whose bottom entry is of the form $(\alpha \wedge \beta)$.

$\vee$-introduction: if P is a proof with conclusion α, and β is a wff then

$$\frac{P}{(\alpha \vee \beta)} \ (\vee I)$$

and

$$\frac{P}{(\beta \vee \alpha)} \ (\vee I)$$

are proofs.

This rule is implemented by a pair of functions, each with domain the set of pairs (P, β) where P is a tree and β is a wff.

$\vee$-elimination: if P_1 and P_2 are proofs with conclusion γ, and Q is a proof with conclusion $(\alpha \vee \beta)$, then

$$\frac{Q \quad P_1^* \quad P_2^*}{\gamma} \ (\vee E)$$

is a proof, where P_1^* is obtained from P_1 by cancelling occurrences of α in the hypotheses of P_1, and P_2^* is obtained from P_2 by cancelling occurrences of β in the hypotheses of P_2.

This rule is implemented by a function whose domain is triples of trees (Q, P_1, P_2) in which P_1 and P_2 have the same bottom element.

$\rightarrow$-introduction: if P is a proof with conclusion β and α is a wff then

$$\frac{P^*}{\alpha \rightarrow \beta} \ (\rightarrow I)$$

is a proof, where P^* is obtained from P by cancelling occurrences of α among the hypotheses of P.

$\rightarrow$-elimination: if P_1 is a proof with conclusion α, P_2 is a proof with conclusion $(\alpha \rightarrow \beta)$ then

$$\frac{P_1 \quad P_2}{\beta} \ (\rightarrow E)$$

is a proof.

$\neg$-introduction: if P_1 has conclusion β, P_2 has conclusion $(\neg\beta)$ and α is a wff then

$$\frac{P_1^* \quad P_2^*}{(\neg\alpha)} \ (\neg I)$$

is a proof, where P_i^* is obtained from P_i by cancelling occurrences of α among the hypotheses.

$\neg$-elimination: if P_1 is a proof with conclusion β, P_2 is a proof with conclusion $(\neg\beta)$ and γ is a wff then

$$\frac{P_1 \quad P_2}{\gamma} \ (\neg E)$$

is a proof.

Contradiction (Reductio Ad Absurdum): If P is a proof and P has conclusion $(\neg(\neg\beta))$ then

$$\frac{P}{\beta} \ (RAA)$$

is a proof