SKETCHY NOTES FOR WEEK 1 OF BASIC LOGIC

A function F is *n-ary (has arity n)* if its domain is a set of n-tuples and *finitary* if it is n-ary for some n.

If F is n-ary and C is a set, we say that C is *closed under* F if $F(a_0, \dots a_{n-1}) \in C$ for all n-tuples $(a_0, \dots a_{n-1}) \in C^n \cap \mathrm{dom}(F)$.

A basic idea in the course is to form the closure of a set B under functions in some set K of finitary functions, that is to say the least set that contains B and is closed under all functions F in K.

Theorem 1. *If B is a set and K is a set of finitary functions then there is a unique set C such that:*

(1) *$B \subseteq C$.*
(2) *C is closed under F for all $F \in K$.*
(3) *For any set C' with $B \subseteq C'$ and C' closed under F for all $F \in K$, we have $C \subseteq C'$.*

Proof. We show first that there is such a set C and then argue that it is unique.

To construct a suitable C, let $C_0 = B$ and then define

$$C_{n+1} = C_n \cup \{F(a_0, \dots a_{k-1}) : F \in K \text{ is } k\text{-ary and } (a_0, \dots a_{k-1}) \in C_n^k \cap \mathrm{dom}(F)\}.$$

Then we set $C = \bigcup_{n \in \mathbb{N}} C_n$.

Clearly $B = C_0 \subseteq C$. To prove closure we use the fact that the sets C_n are increasing in the sense that $m < n$ implies $C_m \subseteq C_n$. So suppose that $F \in K$ is k-ary and $(a_0, \dots a_{k-1}) \in C^k \cap \mathrm{dom}(F)$. For each $i < k$ we choose n_i such that $a_i \in C_{n_i}$ and then set $n^* = \max\{a_0, \dots a_{k-1}\}$. Since the C_i's are increasing all the a_i are in C_{n^*}, so that $(a_0, \dots a_{k-1}) \in C_{n^*}^k \cap \mathrm{dom}(F)$. By definition $F(a_0, \dots a_{k-1}) \in C_{n^*+1}$, so that $F(a_0, \dots a_{k-1}) \in C$.

For the last property of C, we assume that $B \subseteq C'$ and C' is closed under F for all $F \in K$, and show by induction that $C_n \subseteq C'$ for all n. This is easy: $C_0 = B$ gives us the base case and the assumption that C' is closed under all $F \in K$ gives us the successor step. Since $C = \bigcup_n C_n$, it follows that $C_n \subseteq B$.

At this point we have the existence of a suitable C. To show uniqueness suppose that C_0 and C_1 both have the properties listed above. Since C_1 contains B and is closed under all F in K, $C_0 \subseteq C_1$. Similarly $C_1 \subseteq C_0$ and so $C_1 = C_0$. □

Definition 1. *In the situation of Theorem 1 we write $C = \mathcal{C}(B, K)$ and say that C is the* closure of B under K, *or that (B, K) is an* inductive structure for C.

As the name suggests, inductive structures let us do induction.

Theorem 2. *(Induction for inductive structures)*

Let (B, K) be an inductive structure for C and suppose that P is a property such that

(1) *$P(b)$ holds for all $b \in B$.*
(2) *For all $F \in K$ and all $(a_0, \dots a_{k-1}) \in C^k \cap \mathrm{dom}(F)$, if $P(a_i)$ holds for all i with $0 \le i < k$ then $P(F(a_0, \dots a_{k-1}))$ holds.*

Then $P(c)$ holds for all $c \in C$.

Proof. Use the description of C as the union of C_n from the proof of Theorem 1. An easy induction on n shows that $P(c)$ holds for all $c \in C_n$. $\square$

Proof by induction goes hand-in-hand with definition by recursion. It is natural to try to define a function H on $\mathcal{C}(B,K)$ by specifying H on B and then specifying $H(F(a_0,\dots a_{k-1}))$ in terms of the a_i and $H(a_i)$, but this can go wrong in general. The following definition eliminates the two problems that definition by recursion on $\mathcal{C}(B,K)$ may have.

Definition 2. *An inductive structure (B,K) for C is* uniquely readable *if and only if:*

(1) *For all $b \in B$, $b \neq F(a_0,\dots a_{k-1})$ for all $F \in K$ and $(a_0,\dots a_{k-1}) \in C^k \cap \mathrm{dom}(F)$.*
(2) *For all $c \in C \setminus B$, $c = F(a_0,\dots a_{k-1})$ for a unique $F \in K$ and a unique tuple $(a_0,\dots a_{k-1}) \in C^k \cap \mathrm{dom}(F)$.*

Before stating the general principle of recursion on a uniquely readable structure, we give a substantive example of such a structure.

We will define the class of wff's (well formed formulae) of propositional logic. This is a set of finite strings of symbols from the alphabet

$$\{A_n : n \in \mathbb{N}\} \cup \{(,)\} \cup \{\vee, \wedge, \neg, \rightarrow\}$$

We set up an inductive structure by defining B to be the set of all strings of length 1 of form A_n, and $K = \{F_\vee, F_\neg, F_\wedge, F_\rightarrow\}$ to be a family of functions for generating strings: $F_\vee(s,t) = (s \vee t)$, $F_\wedge(s,t) = (s \wedge t)$, $F_\rightarrow(s,t) = (s \rightarrow t)$, $F_\neg(s) = (\neg s)$. The domin of $F_\vee$ is the set of all pairs of strings from our alphabet, and similarly for the others. The set of wff's is the closure of B under K.

Theorem 3. *The inductive structure we just described for the wff's of propositional logic is a uniquely readable inductive structure.*

Proof. We proceed by a series of easy lemmas, each one is proved by induction on the inductive structure as in Theorem 2.

Lemma 1. *Every wff has an equal number of ('s and)'s.*

Lemma 2. *Every non-empty initial segment of a wff has strictly more ('s than)'s.*

Lemma 3. *An inital segment of a wff is not a wff.*

Lemma 4. *Every wff begins either with A_n for some n or (.*

Now we prove unique readability. It is easy to see that any element A_n of B is not in the range of any $F \in K$. If $s = (\neg t)$ then $s \neq (t' @ u')$ for wffs t, t', u' and $@ \in \{\wedge, \vee, \rightarrow\}$, as the first entry in t' can't be $\neg$. If $s = (t @ u) = (t' @' u')$ for wffs t, t', u, u' and $@, @' \in \{\wedge, \vee, \rightarrow\}$, then each of t and t' can't be an initial segment of the other, so $t = t'$ from which we see $@ = @'$ and $u = u'$. $\square$