21-300 F15 HW 7

IMPORTANT: PLEASE EMAIL COMPLETED HOMEWORKS TO:

`lserafin.algebra@gmail.com`

(1) Let $\mathcal{L}$ be some FOL. A sentence in $\mathcal{L}$ is said to be a $\forall\exists$ sentence if it has the form $\forall y_1 \ldots \forall y_m \exists z_1 \ldots \exists z_n \psi$ where ψ is quantifier free.

Suppose that T is a set of $\forall\exists$ sentences and that $\langle \mathcal{M}_n : n \in \mathbb{N} \rangle$ is a sequence of structures, such that $\mathcal{M}_n \models T$ and $\mathcal{M}_n$ is a substructure of $\mathcal{M}_{n+1}$. Letting $M = \bigcup_n M_n$, we define an $\mathcal{L}$-structure $\mathcal{M}$ with underlying set M in the natural way. Prove that $\mathcal{M} \models T$.

(2) Recall that $\mathcal{M}$ is an *elementary substructure of* $\mathcal{N}$ when $\mathcal{M}$ is a substructure of $\mathcal{N}$, and $\mathcal{M} \models \phi(\vec{m}) \iff \mathcal{N} \models \phi(\vec{m})$ for all formulae $\phi(\vec{x})$ and all tuples $\vec{m}$ of elements of M.

Let $\langle \mathcal{M}_n : n \in \mathbb{N} \rangle$ be a sequence of $\mathcal{L}$-structures, such that $\mathcal{M}_n$ is an elementary substructure of $\mathcal{M}_{n+1}$. Letting $M = \bigcup_n M_n$, we define an $\mathcal{L}$-structure $\mathcal{M}$ with underlying set M in the natural way. Prove that $\mathcal{M}_n$ is an elementary substructure of $\mathcal{M}$ for all n.

Hint: Prove by induction on ϕ that for all n and all $\vec{m}$ from M_n, $\mathcal{M}_n \models \phi(\vec{m}) \iff \mathcal{M} \models \phi(\vec{m})$.

(3) Let $\mathcal{L}$ be a countable FOL, let $\mathcal{M}$ be an infinite $\mathcal{L}$-structure and let Y be any set. Prove that there exists $\mathcal{N}$ such that $\mathcal{M}$ is an elementary substructure of $\mathcal{N}$ and there is an injective function from Y to N.

Hint: Expand the language with constants c_a for $a \in M$ and d_y for $y \in Y$. Show that the theory $S \cup T$ is consistent where S is the complete diagram of $\mathcal{M}$ and T is the set of formulae $d_y \neq d_{y'}$ for $y \neq y'$.