21-300 F15 HW 6

IMPORTANT: PLEASE EMAIL COMPLETED HOMEWORKS TO:

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(1) Let $\mathcal{M}$ be a substructure of $\mathcal{N}$. Suppose that for every formula $\psi(z, y_1, \dots y_n)$ and all $a_1, \dots a_n \in M$, if there is $b \in N$ such that $\mathcal{N} \models \psi(b, a_1, \dots a_n)$ then there is $a \in M$ such that $\mathcal{N} \models \psi(a, a_1, \dots a_n)$.

Prove that $\mathcal{M}$ is an elementary substructure of $\mathcal{N}$, that is to say for all $\phi(z_1, \dots z_n)$ and all $a_1, \dots a_n \in M$, $\mathcal{M} \models \phi(a_1, \dots a_n) \iff \mathcal{N} \models \phi(a_1, \dots a_n)$.

Hint: You know from the midterm (and may assume here) that the conclusion is true for all quantifier free formulae ϕ. Use the assumption to power an induction. Be careful, the conclusion talks about satisfaction in $\mathcal{M}$ and $\mathcal{N}$ but the assumption only mentions satisfaction in $\mathcal{N}$.

(2) Let $\mathcal{L}$ be a language with constant symbols 0 and 1, binary function symbols $+$, $\times$, *pow* (for power) and binary relation symbols $\equiv$ (the equality symbol) and $<$. In a slight abuse of notation, let $\mathbb{R}$ be the $\mathcal{L}$-structure whose underlying set is the real numbers in which each symbol is given the natural interpretation; $pow^{\mathbb{R}}$ is the function which takes (x, y) to x^y when $x > 0$ and to zero otherwise.

Let T be the complete diagram of $\mathbb{R}$, that is the set of all sentences of the expanded language which are true in $\mathbb{R}$.

(a) Let c be a new constant and let T^* be the theory $T \cup \{c_0 < c < c_r : r \in \mathbb{R}, r > 0\}$. Prove that T^* is consistent.

(b) Let $\mathbb{R}^*$ be a model of T^*, and (in another abuse of notation) identify the real number r with the interpretation of c_r in $\mathbb{R}^*$. Let d be the interpretation of c in $\mathbb{R}^*$.

(i) Prove that in $\mathbb{R}^*$ there is a unique element d' such that $d \times d' = 1$.

(ii) Prove that in $\mathbb{R}^*$, $d' > n$ for every natural number n.

(iii) Prove that in $\mathbb{R}^*$ there is an element e such that $0 < e < d^n$ for every natural number n.

(3) Let ϕ_n be a sentence in the language of graphs expressing "for every pair (A, B) of sets of vertices with $A \cap B = \emptyset$ and $|A| = |B| = n$ there is a vertex $v \notin A \cup B$ such that vEw for all $w \in A$ and $\neg vEw$ for all $w \in B$". Let $T^* = T_{graphs} \cup \{\phi_n : n > 0\}$.

(a) Prove that T^* has at least one countably infinite model (harder than it looks, try making a model where the underlying set is $\mathbb{N}$ using a listing of all pairs (A, B) of disjoint subsets of $\mathbb{N}$ which have the same size).

(b) Prove that any two countably infinite models of T^* are isomorphic. Hint: back and forth.

(4) Let T be a theory and let $T \vdash \psi$, where ψ is a quantifier-free sentence containing some constant symbols $c_1, \dots c_n$ that are distinct and do not appear in T. Prove that $T \vdash \forall y_1 \ \dots \forall y_n \ \psi'$, where $y_1, \dots y_n$ are distinct

variable symbols not appearing in ψ and ψ' is the result of replacing c_i by y_i for each i.

(5) Let T be a theory in a first order language $\mathcal{L}$, and let $T_\forall$ be the set of universal sentences ψ of $\mathcal{L}$ such that $T \vdash \psi$ (a sentence is *universal* if it has the form $\forall y_1 \ldots \forall y_n\, \psi(y_1, \ldots y_n)$ where ψ is quantifier-free). Prove that if $\mathcal{N} \models T$ and $\mathcal{M}$ is a substructure of $\mathcal{N}$ then $\mathcal{M} \models T_\forall$.

(6) Let T be a theory and let $\mathcal{M} \models T_\forall$. Let $T^* = T \cup D$ where D is the "atomic diagram" of $\mathcal{M}$, that is the set of all atomic sentences ψ in the expanded langauge for $\mathcal{M}$ such that $\mathcal{M} \models \psi$.

Prove that T^* is consistent. Hint: If not then T proves the negation of a finite conjunction of elements of D. Now use a previous question.

Prove there is a structure $\mathcal{N}$ such that $\mathcal{N} \models T$ and $\mathcal{M}$ is a substructure of $\mathcal{N}$.