21-300 F15 HW 5-6-7 SOLUTIONS

Homework 5

(1) (20 points total) Let T be a first-order theory in some language $\mathcal{L}$, let ϕ be a formula with x as its only free variable. Use the Completeness and Soundness theorems to show that

(a) (10 points) If $T \vdash \phi[x/c]$ where c is a constant symbol not appearing in T or ϕ, then $T \vdash \forall x\ \phi$.

(b) (10 points) If $T \cup \{\phi[x/c]\} \vdash \psi$, where c is a constant symbol not appearing in $T \cup \{\phi, \psi\}$, then $T \cup \{\exists x\ \phi\} \vdash \psi$.

NOTE: You can do this one by direct manipulation of proofs, replacing c by a suitable variable symbol. This is OK as long as you are careful.

The key points are that by completeness and soundness, plus considerations about reducts and expansions:

- $T \vdash \psi$ if and only if $T \models \psi$.
- We do not need to be careful about exactly which language we are working in when we write $T \vdash \psi$, as long as the set of formulae of the language contains $T \cup \{\psi\}$.

For the first part, suppose that $T \vdash \phi[x/c]$ with c not appearing in T or ϕ, and note that since c does not appear in ϕ every appearance of c in $\phi[x/c]$ corresponds to an appearance of x in ϕ.

We want to show that $T \vdash \forall x\ \phi$, and it will be enough to show that $T \models \forall x\ \phi$.

By hypothesis: whenever $\mathcal{M}$ is a structure for a language whose formulae include $T \cup \{\phi[x/c]\}$, and $\mathcal{M} \models T$, then $\mathcal{M} \models \phi[x/c]$. The second key point is that since c does not appear in T, we can vary the interpretation of c and still obtain a model of T.

Since c has no appearances in ϕ, it follows that if $\mathcal{M} \models T$ then $\mathcal{M} \models \phi[x/c_a]$ for every $a \in M$, that is $\mathcal{M} \models \forall x\ \phi$.

The argument for the second part is quite similar. Let $\mathcal{M} \models T \cup \{\exists x\ \phi\}$. We may as well assume that $\mathcal{M}$ is a structure for a language that does not include c, and then expand it to a structure for a language with c where c is interpreted as some a such that $\mathcal{M} \models \phi[x/c_a]$. Then $\mathcal{M} \models \phi[x/c]$, so $\mathcal{M} \models \psi$ and we are done.

(2) (25 points total) Let $\mathcal{M}$ and $\mathcal{N}$ be structures for some first-order language $\mathcal{L}$ with underlying sets M and N respectively. A function $f : M \longrightarrow N$ is called a *homomorphism from $\mathcal{M}$ to $\mathcal{N}$* if

(a) $f(c^{\mathcal{M}}) = c^{\mathcal{N}}$ for all constant symbols c.

(b) For every relation symbol R with arity k and every $a_1, \dots a_k \in M$, $R^{\mathcal{M}}(a_1, \dots a_k) \implies R^{\mathcal{N}}(f(a_1), \dots f(a_k))$.

(c) For every function symbol G with arity l and every $a_1, \dots a_l \in M$, $f(G^{\mathcal{M}}(a_1, \dots a_l)) = G^{\mathcal{N}}(f(a_1), \dots f(a_l))$.

Let f be a homomorphism from $\mathcal{M}$ to $\mathcal{N}$. For each closed term t of the expanded language for $\mathcal{M}$, let $\bar{t}$ be the closed term of the expanded language for $\mathcal{N}$ obtained by replacing each constant c_a by the constant $c_{f(a)}$.

(a) (10 points) Prove that $f(t^{\mathcal{M}}) = \bar{t}^{\mathcal{N}}$ for each closed term t of the expanded language for $\mathcal{M}$.
If t is a constant symbol c of the original language, $t = \bar{t} = c$. $f(c^{\mathcal{M}}) = c^{\mathcal{N}}$ by the definition of homomorphism.
If t is a constant symbol c_a for $a \in M$, then $\bar{t} = c_{f(a)}$, and $f(c_a^{\mathcal{M}}) = f(a) = c_{f(a)}^{\mathcal{N}}$.
If t is $g(t_1, \dots t_m)$ then $\bar{t}$ is $g(\bar{t}_1, \dots \bar{t}_m)$, and

$$f(g^{\mathcal{M}}(t_1^{\mathcal{M}}, \dots t_m^{\mathcal{M}})) = g^{\mathcal{N}}(f(t_1^{\mathcal{M}}), \dots f(t_m^{\mathcal{M}})) = g^{\mathcal{N}}(t_1^{\mathcal{N}}, \dots t_m^{\mathcal{N}})$$

by induction and the definition of homomorphism.

(b) (15 points) Prove (by induction on ϕ) that if $\phi(y_1, \dots y_n)$ is a formula of $\mathcal{L}$ involving only the connectives $\wedge$ and $\vee$ and only the quantifier $\exists$, then for all $a_1 \dots a_n \in M$ $\mathcal{M} \models \phi(a_1, \dots a_n) \implies \mathcal{N} \models \phi(f(a_1), \dots f(a_n))$.
Let ϕ be an atomic formula $R(t_1, \dots t_m)$ and let $y_1, \dots y_n$ be the variables appearing in the terms t_i. Let T be the substitution which replaces y_i by c_{a_i}. Now if $\mathcal{M} \models \phi(a_1, \dots a_n)$, then by definition $R^{\mathcal{M}}((t_1[T])^{\mathcal{M}}, \dots (t_m[T])^{\mathcal{M}})$. Since f is a homomorphism, it follows from the last part that $R^{\mathcal{N}}(\overline{t_1[T]}^{\mathcal{N}}, \dots \overline{t_m[T]}^{\mathcal{N}})$.
Now $\overline{t_i[T]} = t_i[U]$, where U is the substitution which replaces y_i by $c_{f(a_i)}$, so $R^{\mathcal{N}}((t_1[U])^{\mathcal{N}}, \dots (t_m[U])^{\mathcal{N}})$, so $\mathcal{N} \models \phi(f(a_1), \dots f(a_n))$.
The inductions steps for $\wedge$, $\vee$ are easy.
Luke: 4 points for this bit.
Finally, suppose ϕ is $\exists z\ \psi(z, y_1, \dots y_m)$ and $\mathcal{M} \models \phi(a_1, \dots a_m)$, so that by definition $\mathcal{M} \models \psi(b, a_1, \dots a_m)$. By induction $\mathcal{N} \models \psi(f(b), f(a_1), \dots f(a_m))$, so that by definition $\mathcal{N} \models \phi(f(a_1), \dots f(a_m))$.

Homework 6

(1) (15 points total) Let $\mathcal{M}$ be a substructure of $\mathcal{N}$. Suppose that for every formula $\psi(z, y_1, \dots y_n)$ and all $a_1, \dots a_n \in M$, if there is $b \in N$ such that $\mathcal{N} \models \psi(b, a_1, \dots a_n)$ then there is $a \in M$ such that $\mathcal{N} \models \psi(a, a_1, \dots a_n)$.

Prove that $\mathcal{M}$ is an elementary substructure of $\mathcal{N}$, that is to say for all $\phi(z_1, \dots z_n)$ and all $a_1, \dots a_n \in M$, $\mathcal{M} \models \phi(a_1, \dots a_n) \iff \mathcal{N} \models \phi(a_1, \dots a_n)$.

Hint: You know from the midterm (and may assume here) that the conclusion is true for all quantifier free formulae ϕ. Use the assumption to power an induction. Be careful, the conclusion talks about satisfaction in $\mathcal{M}$ and $\mathcal{N}$ but the assumption only mentions satisfaction in $\mathcal{N}$.

Taking the hint, we proceed by induction on the construction of the formula ϕ. The connective cases are easy, so we concentrate on the quantifier cases.

$\exists$ case: ϕ is of the form $\exists y\ \psi(y, z_1, \dots z_n)$, where the IH tells us that for all $b, a_1, \dots a_n \in M$, $\mathcal{M} \models \psi(b, a_1, \dots a_n) \iff \mathcal{N} \models \psi(b, a_1, \dots a_n)$.

If $\mathcal{M} \models \phi(a_1, \dots a_n)$, then by definition $\mathcal{M} \models \psi(b, a_1, \dots a_n)$ for some $b \in M$, by the IH $\mathcal{N} \models \psi(b, a_1, \dots a_n)$, and by definition again $\mathcal{N} \models \phi(a_1, \dots a_n)$. If $\mathcal{N} \models \phi(a_1, \dots a_n)$, then by definition $\mathcal{N} \models \psi(c, a_1, \dots a_n)$ for some $c \in N$, by the assumption $\mathcal{N} \models \psi(b, a_1, \dots a_n)$ for some $b \in M$, by the IH $\mathcal{M} \models \psi(b, a_1, \dots a_n)$, and finally by definition $\mathcal{M} \models \phi(a_1, \dots a_n)$.

$\forall$ case: ϕ is of the form $\forall y\ \psi(y, z_1, \dots z_n)$, where again the IH tells us that for all $b, a_1, \dots a_n \in M$, $\mathcal{M} \models \psi(b, a_1, \dots a_n) \iff \mathcal{N} \models \psi(b, a_1, \dots a_n)$.

Notice that $\mathcal{M} \models \neg\psi(b, a_1, \dots a_n) \iff \mathcal{N} \models \neg\psi(b, a_1, \dots a_n)$. It suffices to show that $\mathcal{M} \models \neg\phi(a_1, \dots a_n) \iff \mathcal{N} \models \neg\phi(a_1, \dots a_n)$, and this is equivalent to showing that $\mathcal{M} \models \exists y\ \neg\psi(y, a_1, \dots a_n) \iff \mathcal{N} \models \exists y\ \neg\psi(y, a_1, \dots a_n)$. Now proceed as in the $\exists$ case.

(2) (20 points total) Let $\mathcal{L}$ be a language with constant symbols 0 and 1, binary function symbols $+$, $\times$, *pow* (for power) and binary relation symbols $\equiv$ (the equality symbol) and $<$. In a slight abuse of notation, let $\mathbb{R}$ be the $\mathcal{L}$-structure whose underlying set is the real numbers in which each symbol is given the natural interpretation; $pow^{\mathbb{R}}$ is the function which takes (x, y) to x^y when $x > 0$ and to zero otherwise.

Let T be the complete diagram of $\mathbb{R}$, that is the set of all sentences of the expanded language which are true in $\mathbb{R}$.

(a) (5 points) Let c be a new constant and let T^* be the theory $T \cup \{c_0 < c < c_r : r \in \mathbb{R}, r > 0\}$. Prove that T^* is consistent.
ETS that for any finite set A of positive reals, $T \cup \{c_0 < c < c_r : r \in A\}$ is consistent. This is easy, interpret the symbols of T inside $\mathbb{R}$ in the standard way and interpret c as something between 0 and $\min(A)$.

(b) Let $\mathbb{R}^*$ be a model of T^*, and (in another abuse of notation) identify the real number r with the interpretation of c_r in $\mathbb{R}^*$. Let d be the interpretation of c in $\mathbb{R}^*$.

(i) (5 points) Prove that in $\mathbb{R}^*$ there is a unique element d' such that $d \times d' = 1$.
Let ψ be the sentence expressing "every element greater than zero has a unique multiplicative inverse". ψ is true in $\mathbb{R}$, hence

$\psi \in T$, hence $\psi \in T^*$. So ψ is true in $\mathbb{R}^*$, $d > 0$ in this structure, and hence there is a unique d' as required.

(ii) (5 points) Prove that in $\mathbb{R}^*$, $d' > n$ for every natural number n.
The sentence expressing that "for all positive x and y, $x > y$ implies $1/x < 1/y$" is true in $\mathbb{R}$ and hence is true in $\mathbb{R}^*$. Of course there is no division symbol in the language, but we can express statements about the unique multiplicative inverse of an element. By construction $0 < d < 1/n$ for all n, so $n < 1/d = d'$.

(iii) (5 points) Prove that in $\mathbb{R}^*$ there is an element e such that $0 < e < d^n$ for every natural number n.
Consider $e = d^{1/d}$, and argue as above that properties of the ordering and the power function justify the conclusion.

(3) (20 points) Let ϕ_n be a sentence in the language of graphs expressing "for every pair (A, B) of sets of vertices with $A \cap B = \emptyset$ and $|A| = |B| = n$ there is a vertex $v \notin A \cup B$ such that vEw for all $w \in A$ and $\neg vEw$ for all $w \in B$". Let $T^* = T_{graphs} \cup \{\phi_n : n > 0\}$.

(a) (10 points) Prove that T^* has at least one countably infinite model (harder than it looks, try making a model where the underlying set is $\mathbb{N}$ using a listing of all pairs (A, B) of disjoint subsets of $\mathbb{N}$ which have the same size).
We will build the graph in stages, at each stage we will label various unordered pairs of integers as either "edges" or "nonedges". Each stage will only label finitely many pairs, and we will arrange that the sets of pairs considered at different stages are disjoint: this will make sure that once we handle a pair it stays handled at subsequent stages, Taking the hint, enumerate all relevant pairs as (A_i, B_i) for $i \in \mathbb{N}$. We will associate to each pair a "witnessing vertex" w_i, making sure that $\max(A_i \cup B_i) < w_i$ and w_i is strictly increasing with i. At stage i, mark all pairs $\{a, w_i\}$ for $a \in A_i$ as edges and all pairs $\{b, w_i\}$ for $b \in B_i$ as non-edges.
Sophisticate's alternative, need some probability theory to see this works: for each $i \neq j$, toss a fair coin to determine if i should be joined to j. For each pair A, B there is $\epsilon > 0$ such that for each $w \notin A \cup B$, the probability that w "works" for A and B is ϵ. Since there are infinitely many w's and the coin tosses are independent, almost surely some w works for A and B. Since there are only countably many pairs, almost surely the random graph we build is as required.

(b) Prove that any two countably infinite models of T^* are isomorphic. Hint: back and forth.
Let X and Y be countably infinite models of T^*. We enumerate them and build an isomorphism basically as we did for DLOWE's. At stage i we have a "partial isomorphism" g_i with domain and range of size $i + 1$. Suppose that i is even (so it is a "forth" stage) and x' is least in the enumeration of X such that $x' \notin \mathrm{dom}(g_i)$. Let $A^* = \{g(x) : x \text{ is joined to } x' \text{ in } X \}$ and $B^* = \{g(x) : x \text{ is not joined to } x' \text{ in } X \}$. use the fact that Y is a model of T to show there is y' joined to all elements of A^* and no elements of B^*, and let $g_{i+1}(x')$ be the least such y'. Use a similar argument for the "back" stages.

(4) (10 points) Let T be a theory and let $T \vdash \psi$, where ψ is a quantifier-free sentence containing some constant symbols $c_1, \dots c_n$ that are distinct and do not appear in T. Prove that $T \vdash \forall y_1 \ \dots \forall y_n \ \psi'$, where $y_1, \dots y_n$ are distinct variable symbols not appearing in ψ and ψ' is the result of replacing c_i by y_i for each i.

It's a routine application of Completeness. We argue that $T \models \forall y_1 \ \dots \forall y_n \ \psi'$, which amounts to showing that if $\mathcal{M} \models T$ then $\mathcal{M} \models \psi'(\vec{a})$ for all $\vec{a} \in M^n$. To see this just expand $\mathcal{M}$ to a structure which interprets c_i as a_i, and use $T \models \psi$.

(5) (5 points) Let T be a theory in a first order language $\mathcal{L}$, and let $T_\forall$ be the set of universal sentences ψ of $\mathcal{L}$ such that $T \vdash \psi$ (a sentence is *universal* if it has the form $\forall y_1 \dots \forall y_n \, \psi(y_1, \dots y_n)$ where ψ is quantifier-free). Prove that if $\mathcal{N} \models T$ and $\mathcal{M}$ is a substructure of $\mathcal{N}$ then $\mathcal{M} \models T_\forall$.

Immediate from a question on the midterm.

(6) (20 points) Let T be a theory and let $\mathcal{M} \models T_\forall$. Let $T^* = T \cup D$ where D is the "atomic diagram" of $\mathcal{M}$, that is the set of all atomic sentences ψ in the expanded language for $\mathcal{M}$ such that $\mathcal{M} \models \psi$.

Prove that T^* is consistent. Hint: If not then T proves the negation of a finite conjunction of elements of D. Now use a previous question.

Taking the hint, if T is inconsistent then $T \vdash \neg\Psi$ where Ψ is a finite disjunction of atomic sentences of the expanded language, each one true in $\mathcal{M}$. Let $a_1, \dots a_n$ be the elements of M such that the corresponding constants appear in Ψ. Since the symbols c_{a_i} don't appear in T, by a preceding question $T \vdash \forall y_1 \ \dots \forall y_n \ \neg\Psi'$ where Ψ' is obtained from Ψ by replacing c_{a_i} by y_i. So $\forall y_1 \ \dots \forall y_n \ \neg\Psi' \in T_\forall$ and hence $\mathcal{M} \models \forall y_1 \ \dots \forall y_n \ \neg\Psi'$, which is a contradiction because $\mathcal{M} \models \Psi'(a_1, \dots a_n)$.

Prove there is a structure $\mathcal{N}$ such that $\mathcal{N} \models T$ and $\mathcal{M}$ is a substructure of $\mathcal{N}$.

A model of T^* is a model of T which contains an isomoprhic copy of $\mathcal{M}$.

Homework 7

(1) (15 points total) Let $\mathcal{L}$ be some FOL. A sentence in $\mathcal{L}$ is said to be a $\forall\exists$ sentence if it has the form $\forall y_1 \ \dots \forall y_m \ \exists z_1 \ \dots \exists z_n \ \psi$ where ψ is quantifier free.

Suppose that T is a set of $\forall\exists$ sentences and that $\langle \mathcal{M}_n : n \in \mathbb{N} \rangle$ is a sequence of structures, such that $\mathcal{M}_n \models T$ and $\mathcal{M}_n$ is a substructure of $\mathcal{M}_{n+1}$. Letting $M = \bigcup_n M_n$, we define an $\mathcal{L}$-structure $\mathcal{M}$ with underlying set M in the natural way. Prove that $\mathcal{M} \models T$.

For each n, $\mathcal{M}_n$ is a substructure of $\mathcal{M}$, so that $\mathcal{M}_n \models \psi(\vec{a}) \iff \mathcal{M} \models \psi(\vec{a})$ for all quantifier-free ψ and all $\vec{a} \in \mathcal{M}_n$.

Now let ϕ be a formula $\forall y_1 \ \dots \forall y_m \ \exists z_1 \ \dots \exists z_n \ \psi$ appearing in T, for some qf ψ. To show that $\mathcal{M} \models \phi$, we fix some $\vec{a} \in M$ and then find t so large that $\vec{a} \in \mathcal{M}_t$.

Since $\mathcal{M}_t \models \phi$, we may find $\vec{b} \in M_t$ such that $\mathcal{M}_t \models \psi(\vec{a}, \vec{b})$. So $\mathcal{M} \models \psi(\vec{a}, \vec{b})$. This completes the verification that $\mathcal{M} \models \phi$.

(2) (20 points) Recall that $\mathcal{M}$ is an *elementary substructure of* $\mathcal{N}$ when $\mathcal{M}$ is a substructure of $\mathcal{N}$, and $\mathcal{M} \models \phi(\vec{m}) \iff \mathcal{N} \models \phi(\vec{m})$ for all formulae $\phi(\vec{x})$ and all tuples $\vec{m}$ of elements of M.

Let $\langle \mathcal{M}_n : n \in \mathbb{N} \rangle$ be a sequence of $\mathcal{L}$-structures, such that $\mathcal{M}_n$ is an elementary substructure of $\mathcal{M}_{n+1}$. Letting $M = \bigcup_n M_n$, we define an $\mathcal{L}$-structure $\mathcal{M}$ with underlying set M in the natural way. Prove that $\mathcal{M}_n$ is an elementary substructure of $\mathcal{M}$ for all n.

Hint: Prove by induction on ϕ that for all n and all $\vec{m}$ from M_n, $\mathcal{M}_n \models \phi(\vec{m}) \iff \mathcal{M} \models \phi(\vec{m})$.

Note that by an easy induction $\mathcal{M}_n$ is an elementary substructure of $\mathcal{M}_{n'}$.

Taking the hint, we start with the observation that the base case (quantifier free ϕ) is easy because $\mathcal{M}_n$ is a substructure of $\mathcal{M}$. The connective steps are also very easy.

Suppose now that $\phi(y_1, \dots y_n)$ has form $\exists x \ \psi(x, y_1, \dots y_n)$ where the IH holds for ψ. If $\mathcal{M}_n \models \phi(\vec{b})$, choose $a \in \mathcal{M}_n$ such that $\mathcal{M}_n \models \psi(a, \vec{b})$, use the IH to conclude that $\mathcal{M} \models \psi(a, \vec{b})$ and hence $\mathcal{M} \models \phi(\vec{b})$.

If $\mathcal{M} \models \phi(\vec{b})$ for some $\vec{b} \in \mathcal{M}_n$, then choose $a \in M$ such that $\mathcal{M} \models \psi(a, \vec{b})$. Choose $n' \geq n$ such that $a \in M_{n'}$. By the IH (for n') $\mathcal{M}_{n'} \models \psi(a, \vec{b})$ and so $\mathcal{M}_{n'} \models \phi(\vec{b})$. Since $\mathcal{M}_n$ is an elementary substructure of $\mathcal{M}_{n'}$, $\mathcal{M}_n \models \phi(\vec{b})$.

Finally we can handle the $\forall$ case by a similar argument, or combine the facts that we did the $\neg$ and $\exists$ steps.

(3) (10 points) Let $\mathcal{L}$ be a countable FOL, let $\mathcal{M}$ be an infinite $\mathcal{L}$-structure and let Y be any set. Prove that there exists $\mathcal{N}$ such that $\mathcal{M}$ is an elementary substructure of $\mathcal{N}$ and there is an injective function from Y to N.

Hint: Expand the language with constants c_a for $a \in M$ and d_y for $y \in Y$. Show that the theory $S \cup T$ is consistent where S is the complete diagram of $\mathcal{M}$ and T is the set of formulae $d_y \neq d_{y'}$ for $y \neq y'$.

If $\mathcal{N} \models S \cup T$ then it contains an elementary substructure isomorphic to $\mathcal{M}$ (which is sufficient, with a bit of relabeling) and the interpretation of the constants d_y gives the required injective function.

Let s_0 be a finite subset of S and t_0 a finite subset of T. $\mathcal{M}$ (viewed as a structure for its expanded language) is a model of s_0, and by choosing distinct elements (which is possible because M is infinite) to interpret those finitely many d_y appearing in t_0 we get an expansion which is a model of $s_0 \cup t_0$.