21-300 F15 HW 5

A note on notation: We often write "$\phi(y_1, \dots y_n)$" to indicate that ϕ is a formula whose free variables have been listed without repetitions as $y_1, \dots y_n$. When $\mathcal{M}$ is a structure for the language of ϕ, and $a_1, \dots a_n \in M$ then we often write "$\mathcal{M} \models \phi(a_1, \dots a_n)$" as shorthand for "$\mathcal{M} \models \phi[y_1/c_{a_1}, \dots y_n/c_{a_n}]$".

IMPORTANT: PLEASE EMAIL COMPLETED HOMEWORKS TO:

`lserafin.algebra@gmail.com`

(1) Let T be a first-order theory in some language $\mathcal{L}$, let ϕ be a formula with x as its only free variable. Use the Completeness and Soundness theorems to show that
   - (a) If $T \vdash \phi[x/c]$ where c is a constant symbol not appearing in T or ϕ, then $T \vdash \forall x\ \phi$.
   - (b) If $T \cup \{\phi[x/c]\} \vdash \psi$, where c is a constant symbol not appearing in $T \cup \{\phi, \psi\}$, then $T \cup \{\exists x\ \phi\} \vdash \psi$.

(2) Let $\mathcal{M}$ and $\mathcal{N}$ be structures for some first-order language $\mathcal{L}$ with underlying sets M and N respectively. A function $f : M \longrightarrow N$ is called a *homomorphism from $\mathcal{M}$ to $\mathcal{N}$* if
   - (a) $f(c^{\mathcal{M}}) = c^{\mathcal{N}}$ for all constant symbols c.
   - (b) For every relation symbol R with arity k and every $a_1, \dots a_k \in M$, $R^{\mathcal{M}}(a_1, \dots a_k) \implies R^{\mathcal{N}}(f(a_1), \dots f(a_k))$.
   - (c) For every function symbol G with arity l and every $a_1, \dots a_l \in M$, $f(G^{\mathcal{M}}(a_1, \dots a_l)) = G^{\mathcal{N}}(f(a_1), \dots f(a_l))$.

   Let f be a homomorphism from $\mathcal{M}$ to $\mathcal{N}$. For each closed term t of the expanded language for $\mathcal{M}$, let $\bar{t}$ be the closed term of the expanded language for $\mathcal{N}$ obtained by replacing each constant c_a by the constant $c_{f(a)}$.
   - (a) Prove that $f(t^{\mathcal{M}}) = \bar{t}^{\mathcal{N}}$ for each closed term t of the expanded language for $\mathcal{M}$.
   - (b) Prove (by induction on ϕ) that if $\phi(y_1, \dots y_n)$ is a formula of $\mathcal{L}$ involving only the connectives $\wedge$ and $\vee$ and only the quantifier $\exists$, then for all $a_1 \dots a_n \in M$ $\mathcal{M} \models \phi(a_1, \dots a_n) \implies \mathcal{N} \models \phi(f(a_1), \dots f(a_n))$.
     Hint: you should find the first part of the question helpful in doing the base case (when ϕ is atomic).