HANDOUT ON PROOFS IN FIRST ORDER LOGIC

We will start with the proof rules for propositional logic (in particular any formula ϕ is a proof with hypothesis ϕ and conclusion ϕ), and add Intro and Elim rules for the quantifiers.

Subtle point: In proofs we reason with formulae, not just sentences.

We define when it is allowed to substitute τ for x in ϕ. Intuitively the idea is that variables appearing in τ should not be bound by quantifiers appearing in ϕ, because this changes the meaning of the formula.

- If ϕ is atomic, τ is allowed for x in ϕ.
- τ is allowed for x in $\neg\psi$ if and only if τ is allowed for x in ψ.
- τ is allowed for x in $\phi @ \psi$ if and only if τ is allowed for x in ϕ and τ is allowed for x in ψ.
- τ is allowed for x in $Qy\psi$ if and only if EITHER x has no free appearances in $Qy\psi$, OR τ is allowed for x in ψ and y does not appear in τ.

Note that: By an easy induction, x is always allowed for x in ϕ.

The rules:

- $\forall$-elimination: If P is a proof with conclusion $\forall x\phi$ and τ is allowed for x in ϕ, then
$$\frac{P}{\phi[x/\tau]}$$
is a proof.
- $\forall$-introduction: If P is a proof with conclusion ϕ and x has no free appearances in the hypotheses of P, then
$$\frac{P}{\forall x\ \phi}$$
is a proof.
- $\exists$-introduction: If P is a proof with conclusion $\phi[x/\tau]$ where τ is allowed for x in ϕ, then
$$\frac{P}{\exists x\ \phi}$$
is a proof.
- $\exists$-elimination: If P is a proof with conclusion $\exists x\ \phi$, and Q is a proof with conclusion ψ where x has no free appearance in ψ and the only free appaearances of x among the hypotheses of Q are in instances of ϕ, then
$$\frac{P \quad Q^*}{\psi}$$
is a proof where Q^* is obtained from Q by cancelling appearances of ϕ in the hypotheses.

Since x is always allowed by x in ϕ, the following are valid proofs:

$$\frac{\forall x\ \phi}{\phi}$$

and

$$\frac{\phi}{\exists x\ \phi}$$

Now for some cautionary examples of broken "proofs" where we get logically invalid results by being careless about the rules:

- Consider the "proof"

$$\frac{\forall x_0\ \exists x_1\ R(x_0, x_1)}{\exists x_1\ R(x_1, x_1)}$$

obtained by misapplying the $\forall E$ rule. It is logically invalid, consider for example the $<$ relation on $\mathbb{N}$. Formally the problem is that x_1 is not allowed for x_0 in $\exists x_1\ R(x_0, x_1)$.

- Another cautionary example of a "proof" that one could build by misapplying $\forall I$ rule:

$$\frac{R(x)}{\forall x\ R(x)}$$

This is clearly silly, being given an example of an x with property R should not let us conclude that everything has property R. Formally the problem is that x has a free appearance in the hypotheses of the 1-line proof $R(x)$.

- Finally two examples to show that both the restrictions in the $\exists E$ rule are important.

The following is a valid proof.

$$\dfrac{\dfrac{R(x) \quad Q(x)}{R(x) \wedge Q(x)}}{\exists x\ R(x) \wedge Q(x)}$$

The following "proof" has been built by misapplying the $\exists E$ rule (on the first application, when we cancelled the hypothesis $R(x)$):

$$\dfrac{\exists x\ Q(x) \qquad \dfrac{\exists x\ R(x) \qquad \dfrac{\dfrac{\cancel{R(x)} \quad \cancel{Q(x)}}{R(x) \wedge Q(x)}}{\exists x\ R(x) \wedge Q(x)}}{\exists x\ R(x) \wedge Q(x)}}{\exists x\ R(x) \wedge Q(x)}$$

This is clearly not valid, consider the integers with the predicates for even and odd. The problem was in the first application of the $\forall E$ rule to the proofs $\exists x\ R(x)$ and

$$\dfrac{\dfrac{R(x) \quad Q(x)}{R(x) \wedge Q(x)}}{\exists x\ R(x) \wedge Q(x),}$$

which is not OK because x appears freely in the hypothesis $Q(x)$.

Also consider the following broken "proof":

$$\dfrac{\dfrac{\exists x\ R(x) \quad \cancel{R(x)}}{R(x)}}{\forall x\ R(x)}$$

Here the problem is the free appearance of x in the conclusion of the one-line proof $R(x)$.