Conditional Probability

Suppose $A \subseteq \Omega$. We define an *induced* probability $\mathbf{P}_A$ by

$$\mathbf{P}_A(\omega) = \frac{\mathbf{P}(\omega)}{\mathbf{P}(A)} \qquad \text{for } \omega \in A.$$

Two Coins
$A = \{\text{At least one coin is a Head}\}$. $\mathbf{P}(A) = 3/4$.

$B = \{\text{Both are heads}\}$. $P(B) = 1/4$.
So $\mathbf{P}_A(B) = 1/3$.

Usually write $\mathbf{P}(B \mid A)$ for $P_A(B)$.

If B is an arbitrary subset of Ω we write

$$\mathbf{P}(B \mid A) = \mathbf{P}_A(A \cap B) = \frac{\mathbf{P}(A \cap B)}{\mathbf{P}(A)}.$$

Two Dice
$A = \{x_1 + x_2 \text{ is odd}\}$. $\mathbf{P}(A) = 1/2$
$B = \{x_1 > x_2\}$. $\mathbf{P}(B) = 5/12$.
$\mathbf{P}(A \cap B) = 1/4$.

$$\mathbf{P}(B \mid A) = 1/2 > \mathbf{P}(B).$$

Binomial n coin tosses. $p = \mathbf{P}(Heads)$ for each toss.
$\Omega = \{H, T\}^n$.

$$\mathbf{P}(\omega) = p^k(1-p)^{n-k}$$

where k is the number of H's in ω.
E.g. $\mathbf{P}(HHTTHTHHTHHTHT) = p^8(1-p)^6$.

Fix k. $A = \{\omega : H \text{ appears } k \text{ times}\}$
$\mathbf{P}(A) = \binom{n}{k} p^k (1-p)^{n-k}$.
If $\omega \in A$ then

$$\mathbf{P}_A(\omega) = \frac{p^k(1-p)^{n-k}}{\binom{n}{k}p^k(1-p)^{n-k}} = \frac{1}{\binom{n}{k}}$$

i.e. *conditional* on there being k heads, each sequence with k heads is equally likely.

Law of Total Probability

Let $B_1, B_2, \ldots, B_n$ be pairwise disjoint events which partition Ω. For any other event A,

$$\mathbf{P}(A) = \sum_{i=1}^{n} \mathbf{P}(A \mid B_i)\mathbf{P}(B_i).$$

Proof

$$\begin{aligned} \sum_{i=1}^{n} \mathbf{P}(A \mid B_i)\mathbf{P}(B_i) &= \sum_{i=1}^{n} \mathbf{P}(B_i \cap A) \\ &= \mathbf{P}\Big(\bigcup_{i=1}^{n} (B_i \cap A)\Big) \quad (1) \\ &= \mathbf{P}(A). \end{aligned}$$

There is equality in (1) because the events $B_i \cap A$ are pairwise disjoint.

Suppose we choose 2 numbers X_1, X_2 randomly from $[n]$. What is the probability of $A = \{|X_1 - X_2| \leq 1\}$?

Let $B_i = \{X_1 = i\}$ for $i \in [n]$. Then

$$\begin{aligned}
\mathbf{P}(A) &= \sum_{i=1}^{n} \mathbf{P}(A \mid B_i)\mathbf{P}(B_i) \\
&= \frac{1}{n}\sum_{i=1}^{n} \mathbf{P}(A \mid B_i) \\
&= \frac{1}{n}\Big(\mathbf{P}(A \mid B_1) + \mathbf{P}(A \mid B_n) + \\
&\quad \sum_{i=2}^{n-1} \mathbf{P}(A \mid B_i)\Big) \\
&= \frac{1}{n}\left(\frac{2}{n} + \frac{2}{n} + \sum_{i=2}^{n-1}\frac{3}{n}\right) \\
&= \frac{3n-2}{n^2}.
\end{aligned}$$

2 sets $S, T \subseteq [n]$ are chosen (i) independently and (ii) uniformly at random from all possible sets. ($\Omega = \{0,1\}^{2n}$). Let

$$A = \{|S| = |T| \text{ and } S \cap T = \emptyset\}.$$

For each $X \subseteq [n]$ we let $B_X = \{S = X\} = \{(X,T) : T \subseteq [n]\}$. Thus for each X, $\mathbf{P}(B_X) = 2^{-n}$. So,

$$\begin{aligned}
\mathbf{P}(A) &= \sum_X \mathbf{P}(A \mid B_X)\mathbf{P}(B_X) \\
&= 2^{-n}\sum_X \binom{n-|X|}{|X|} 2^{-n} \qquad (2) \\
&= 4^{-n}\sum_{k=0}^{n}\binom{n}{k}\binom{n-k}{k}.
\end{aligned}$$

(2) follows from the fact that there are $\binom{n-|X|}{|X|}$ subsets of the same size as X which are disjoint from X.

Independence

Two events A, B are said to be *independent* if

$$\mathbf{P}(A \cap B) = \mathbf{P}(A)\mathbf{P}(B),$$

or equivalently

$$\mathbf{P}(A \mid B) = \mathbf{P}(A).$$

(i) **Two Dice**
$A = \{\omega : x_1 \text{ is odd}\}$, $B = \{\omega : x_1 = x_2\}$.
$|A|$=18, $|B|$=6, $|A \cap B|$=3.
$\mathbf{P}(A) = 1/2$, $\mathbf{P}(B) = 1/6$, $\mathbf{P}(A \cap B) = 1/12$.
A, B are independent.

(ii) $A = \{x_1 \geq 3\}$, $B = \{x_1 \geq x_2\}$.
$|A|$=24, $|B|$=21, $|A \cap B|$=18.
$\mathbf{P}(A) = 2/3$, $\mathbf{P}(B) = 7/12$, $\mathbf{P}(A \cap B) = 1/2$.
A, B are not independent.

Random Bits

Suppose $\Omega = \{0,1\}^n = \{(x_1, x_2, \ldots, x_n) : x_j = 0/1\}$ with uniform distribution.
Suppose event A is determined by the values of x_i, $i \in \Delta_A$
e.g. if $A = \{x_1 = x_2 = \cdots = x_{10} = 0\}$ then $\Delta_A = \{1, 2, \ldots, 10\}$.

More Precisely: for $S \subseteq [n]$ and $x \in \Omega$ let $x_S \in \{0,1\}^S$ be defined by $(x_S)_i = x_i$, $i \in S$.
Ex. $n = 10$, $S = \{2,5,8\}$ and
$x = (0,0,1,0,0,1,1,1,1,0)$. $x_S = \{0,0,1\}$.

A is *determined* by Δ_A if $\exists S_A \subseteq \{0,1\}^{\Delta_A}$ such that $x \in A$ iff $x_{\Delta_A} \in S_A$. Furthermore, no subset of Δ_A has this property.

In our example above,
$S_A = \{(0,0,0,0,0,0,0,0,0,0)\}$ – ($|S_A| = 1$ here.)

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Claim: if events A, B are such that $\Delta_A \cap \Delta_B = \emptyset$ then A and B are independent.

$$\mathbf{P}(A) = \frac{|S_A|}{2^{|\Delta_A|}} \text{ and } \mathbf{P}(B) = \frac{|S_B|}{2^{|\Delta_B|}}.$$

$$\begin{aligned}
\mathbf{P}(A \cap B) &= \frac{1}{2^n} \sum_{x \in \{0,1\}^n} 1_{\{x_{\Delta_A} \in S_A, x_{\Delta_B} \in S_B\}} \\
&= \frac{1}{2^n} |S_A| \, |S_B| 2^{n-|I_A|-|I_B|} \\
&= \mathbf{P}(A)\mathbf{P}(B).
\end{aligned}$$