Discrete Probability

Ω is a finite or countable set – called the *Probability Space*

$\mathbf{P} : \Omega \to \mathbf{R}^+$.

$\sum_{\omega \in \Omega} \mathbf{P}(\omega) = 1$.

If $\omega \in \Omega$ then $\mathbf{P}(\omega)$ is the *probability* of ω.

Fair Coin
$\Omega = \{H, T\}$, $\mathbf{P}(H) = \mathbf{P}(T) = 1/2$.

Dice
$\Omega = \{1, 2, \ldots, 6\}$, $\mathbf{P}(i) = 1/6$, $1 \leq i \leq 6$.

Both are examples of a *uniform distribution*:

$$\mathbf{P}(\omega) = \frac{1}{|\Omega|} \qquad \forall \omega \in \Omega.$$

Geometric or number of Bernouilli trials until success
$\Omega = \{1, 2, \ldots, \}$, $\mathbf{P}(k) = (1-p)^{k-1}p, \quad k \in \Omega$.

Repeat "experiment" until success – k is the total number of trials.
p is the probability of success.

$\mathbf{P}(S) = p, \mathbf{P}(FS) = p(1-p)$,
$\mathbf{P}(FFS) = p^2(1-p), \mathbf{P}(FFFS) = (1-p)^3p \ldots,$.

Note that

$$\sum_{k=0}^{\infty} (1-p)^{k-1}p = \frac{p}{1-(1-p)} = 1.$$

Roll Two Dice

Probability Space 1: $\Omega = [6]^2 = \{(x_1, x_2) : 1 \leq x_1, x_2 \leq 6\}$
$\mathbf{P}(x_1, x_2) = 1/36$ for all x_1, x_2.

Probability Space 2: $\Omega = \{2, 3, 4, , \ldots, 12\}$
$\mathbf{P}(2) = 1/36$, $\mathbf{P}(3) = 2/36$, $\mathbf{P}(4) = 3/36$, $\ldots, \mathbf{P}(12) = 1/36$.

Events

$A \subseteq \Omega$ is called an *event*.

$$\mathbf{P}(A) = \sum_{\omega \in A} \mathbf{P}(\omega).$$

(i) **Two Dice**

$A = \{x_1 + x_2 = 7\}$ where x_i is the value of dice i.
$A = \{(1,6),(2,5),\ldots,(6,1)\}$ and so $\mathbf{P}(A) = 1/6$.

(ii) **Pennsylvania Lottery**

Choose 7 numbers I from [80]. State randomly chooses $J \subseteq [80]$, $|J| = 11$.

$$WIN = \{J : \ J \supseteq I\}.$$

$\Omega = \{$11 element subsets of [80]$\}$ with uniform distribution.
$|WIN|$ = no. subsets which contain I – $\binom{73}{4}$.

$$\begin{aligned}\mathbf{P}(WIN) &= \frac{\binom{73}{4}}{\binom{80}{11}} = \frac{\binom{11}{7}}{\binom{80}{7}} \\ &= \frac{9}{86637720} \approx \frac{1}{9,626,413}.\end{aligned}$$

Poker

Choose 5 cards at random. $|\Omega| = \binom{52}{5}$, uniform distribution.

(i) **Triple** – 3 cards of same value e.g. Q,Q,Q,7,5.
$\mathbf{P}(\text{Triple}) = (13 \times 4 \times 48 \times 44/(2\binom{52}{5})) \approx .021$.

(ii) **Full House** – triple plus pair e.g. J,J,J,7,7.
$\mathbf{P}(\text{FullHouse}) = (13 \times 4 \times 12 \times 6)/\binom{52}{5} \approx .007$.

(iii) **Four of kind** – e.g. 9,9,9,9,J.
$\mathbf{P}(\text{Four of Kind}) = (13 \times 48)/\binom{52}{5} = 1/16660$.

Birthday Paradox

$\Omega = [n]^k$ – uniform distribution, $|\Omega| = n^k$.
$D = \{\omega \in \Omega;$ symbols are distinct$\}$.

$$\mathbf{P}(D) = \frac{n(n-1)(n-2)\ldots(n-k+1)}{n^k}.$$

$n = 365, k = 26$ – birthdays of 26 randomly chosen people.
$\mathbf{P}(D) < .5$ i.e. probability 26 randomly chosen people have distinct birthdays is ¡ .5. (Assumes people are born on random days).

Reliability through redundancy: space ship has 7 independent on board computers.

The navigational decisions are reached by a majority vote of the seven computers.

If each computer is correct with probability $p = .99$, what is the probability the system gives a correct answer?

$$\sum_{i=4}^{7} \binom{7}{i} p^i (1-p)^{7-i} = .9999996583\ldots \ .$$