Pigeon-hole principle

A function $f : A \to X$ is 1-1 (or an *injection*) if whenever $x \neq y$ then $f(x) \neq f(y)$.

Theorem 1 *If $A = \{a_1, a_2, \ldots, a_r\}$ then the number of injections from A to X is $p(n, r)$. For $r > n$ it is 0.*

Proof To each function $f : A \to X$ we associate the sequence $b_1, b_2, \ldots, b_r$ where $b_i = f(a_i)$, $i = 1, 2, \ldots, r$. f is an injection iff the sequence $b_1 b_2 \cdots b_r$ has no repetitions. □

The (trivial) case $r > n$ is called the *pigeon-hole principle*.

Informally: If more than n pigeons are to be placed in n pigeon-holes, at least one hole will end up with more than one pigeon. (For $r > n$ the number of injections from A to X is 0.)

Example 1: There are at least 10^4 people in China who have exactly the same number of strands of hair.

Proof. A human may have $0 \leq x \leq 10^5 - 1$ hair strands. There are more then 10^9 people living in China. Label a 'hole' by the number of hair strands and put a person in hole i if she/he has exactly i hair strands. There must be at least one hole with 10^4 people or more people. Indeed, assume to the contrary, that this is not true. Then the total the number of people in China is at most $10^4 \times 10^5 = 10^9$, a contradiction. □

Observe, that we used a more general version of the *pigeon-hole principle*, which informally says the following. If more than nk pigeons are to be placed in n pigeon-holes, one hole will end up with more than k pigeons.

Example 2. Positive integers n and k are co-prime if their largest common divisor is 1. If we take an arbitrary subset A of $n+1$ integers from the set $[2n] = \{1, \dots n\}$ it will contain a pair of co-prime integers.

If we take the n even integers between 1 and $2n$. This set of n elements does not contain a pair of mutually prime integers. Thus we cannot replace the $n+1$ by n in the statement. We say that the statement is *tight*.

Define the holes as sets $\{1, 2\}, \{3, 4\}, \dots \{2n-1, 2n\}$. Thus n holes are defined. If we place the $n+1$ integers of A into their corresponding holes – by the pigeon-hole principle – there will be a hole, which will contains two numbers. This means, that A has to contain two consecutive integers, say, x and $x+1$. But two such numbers are always co-prime. If some integer $y \neq 1$ divides x, i.e., $x = ky$, then $x+1 = ky+1$ and this is not divisible by y. □

Theorem 2 *(Erdős-Szekeres) An arbitrary sequence of integers* $(a_1, a_2, \dots, a_{k^2+1})$ *contains a monotone subsequence of length* $k+1$.

Proof. Let $(a_i, a_i^1, a_i^2, \dots, a_i^{\ell-1})$ be the longest *monotone increasing* subsequence of $(a_1, \dots, a_{k^2+1})$ that starts with a_i $(1 \leq i \leq k^2+1)$, and let $\ell(a_i)$ be its length.

If for some $1 \leq i \leq k^2+1$, $\ell(a_i) \geq k+1$, then $(a_i, a_i^1, a_i^2, \dots, a_i^{l-1})$ is a monotone increasing subsequence of length $\geq k+1$.

So assume that $\ell(a_i) \leq k$ holds for every $1 \leq i \leq k^2+1$.

Label a 'hole' by the length of a sequence and place the *monotone increasing* subsequence $(a_i, a_i^1, a_i^2, \dots, a_i^{\ell-1})$ into the hole ℓ, which is its length. There are k^2+1 subsequences and k non-empty holes (different lengths), so by the pigeon-hole principle there will be (at least) $k+1$ sequences of the same length ℓ^*. Let $a_{i_1}, a_{i_2}, \dots, a_{i_{k+1}}$ be the first entries of these sequences. Then $a_{i_1} \geq a_{i_2} \geq \dots \geq a_{i_{k+1}}$ holds. Indeed, assume to the contrary that $a_{i_m} \leq a_{i_n}$ for some $1 \leq m < n \leq k+1$. Then $a_{i_m} \leq a_{i_n} \leq a_{i_n}^1 \leq a_{i_n}^2 \leq \dots \leq a_{i_n}^{\ell^*-1}$, i.e., $\ell(a_{i_m}) \geq \ell^*+1$, a contradiction. □