Grid path problems

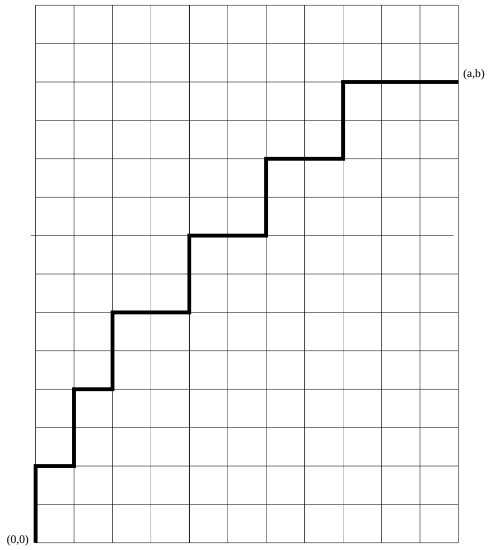
A *monotone path* is made up of segments $(x,y) \to (x+1,y)$ or $(x,y) \to (x,y+1)$.

PATHS$((a,b) \to (c,d))$={monotone paths from (a,b) to (c,d)}.
We drop the $(a,b) \to$ for paths starting at (0,0).
We consider 3 questions: Assume $a, b \geq 0$.

1. How large is PATHS(a,b)?

2. Assume $a < b$. Let PATHS$_>(a,b)$ be the set of paths in PATHS(a,b) which do not touch the line $x = y$ except at (0,0). How big is PATHS$_>(a,b)$?

3. Assume $a \leq b$. Let PATHS$_\geq(a,b)$ be the set of paths in PATHS(a,b) which do not pass through points with $x > y$. How big is PATHS$_\geq(a,b)$?



1. STRINGS(a,b) = $\{x \in \{\mathrm{R},\mathrm{U}\}^* :$ x has a R's and b U's$\}$. *

Natural bijection between PATHS(a,b) and STRINGS(a,b):
Path moves to Right, add R to sequence.
Path goes up, add U to sequence.

So
$$|\text{PATHS}(a,b)| = |\text{STRINGS}(a,b)| = \binom{a+b}{a}$$
since to define a string we have state which of the $a+b$ places contains an R.

*$\{\mathrm{R},\mathrm{U}\}^*$ = set of strings of R's and U's

2. Every path in PATHS$_>(a,b)$ goes through (0,1). So

$$|\text{PATHS}_>(a,b)| = |\text{PATHS}((0,1) \to (a,b))| \\ -|\text{PATHS}_{\not>}((0,1) \to (a,b))|.$$

Now
$$|\text{PATHS}((0,1) \to (a,b))| = \binom{a+b-1}{a}$$
and

$$\text{PATHS}_{\not>}((0,1) \to (a,b))| = \\ |\text{PATHS}((1,0) \to (a,b))| = \binom{a+b-1}{a-1}.$$

We explain the first equality momentarily. Thus
$$|\text{PATHS}_>(a,b)| = \binom{a+b-1}{a} - \binom{a+b-1}{a-1} \\ = \frac{b-a}{a+b}\binom{a+b}{a}.$$

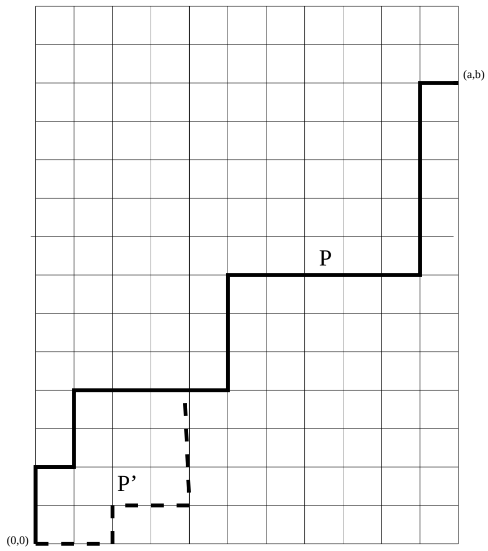
Suppose $P \in \mathrm{PATHS}_{\not>}((0,1) \to (a,b))$. We define $P' \in \mathrm{PATHS}((1,0) \to (a,b))$ in such a way that

$$P \to P' \text{ is a bijection.}$$

Let (c,c) be the first point of P, which lies on the line $L = \{x = y\}$ and let S denote the initial segment of P going from $(0,1)$ to (c,c).

P' is obtained from P by deleting S and replacing it by its reflection S' in L.

To show that this defines a bijection, observe that if $P' \in \mathrm{PATHS}((1,0) \to (a,b))$
then a similarly defined *reverse reflection* yields a $P \in \mathrm{PATHS}_{\not>}((0,1) \to (a,b))$.



3. Suppose $P \in \mathrm{PATHS}_{\geq}(a,b)$. We define $P" \in \mathrm{PATHS}_{>}(a,b+1)$ in such a way that

$$P \to P" \text{ is a bijection.}$$

Thus

$$|\mathrm{PATHS}_{\geq}(a,b)| = \frac{b-a+1}{a+b+1}\binom{a+b+1}{a}.$$

In particular

$$\begin{aligned} |\mathrm{PATHS}_{\geq}(a,a)| &= \frac{1}{2a+1}\binom{2a+1}{a} \\ &= \frac{1}{a+1}\binom{2a}{a}. \end{aligned}$$

The final expression is called a *Catalan Number*.

The bijection

Given P we obtain $P"$ by *raising it vertically one position and then adding the segment* $(0,0) \to (0,1)$.

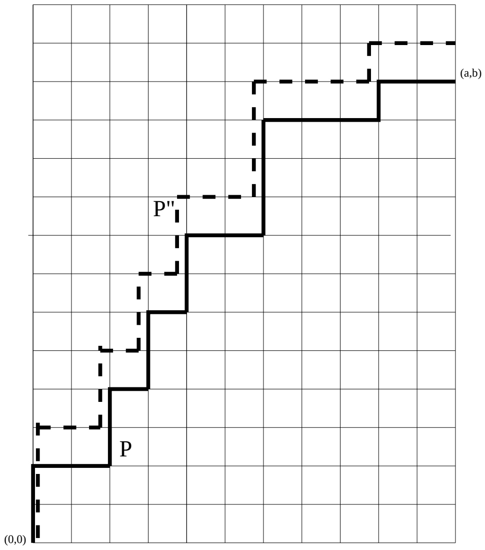
More precisely, if

$$P = (0,0), (x_1,y_1), (x_2,y_2), \ldots, (a,b)$$

then

$$P" = (0,0), (0,1), (x_1, y_1+1), \ldots, (a, b+1).$$

This is clearly a 1-1 onto function between $\mathrm{PATHS}_{\geq}(a,b)$ and $\mathrm{PATHS}_{>}(a,b+1)$.


(a,b)
P''
P
(0,0)