Multi-sets

Suppose we allow elements to appear several times in a set: $\{a, a, a, b, b, c, c, c, d, d\}$.
To avoid confusion with the standard definition of a set we write $\{3 \times a, 2 \times b, 3 \times c, 2 \times d\}$.

How many distinct permutations are there of the multiset $\{a_1 \times 1, a_2 \times 2, \ldots, a_n \times n\}$?

Ex. $\{2 \times a, 3 \times b\}$.

$aabbb$; $ababb$; $abbab$; $abbba$; $baabb$
$babab$; $babba$; $bbaab$; $bbaba$; $bbbaa$.

Start with $\{a_1, a_2, b_1, b_2, b_3\}$ which has $5! = 120$ permutations:
$\ldots a_2 b_3 a_1 b_2 b_1 \ldots a_1 b_2 a_2 b_1 b_3 \ldots$
After erasing the subscripts each possible sequence e.g. $ababb$ occurs $2! \times 3!$ times and so the number of permutations is $5!/2!3! = 10$.

In general if $m = a_1 + a_2 + \cdots + a_n$ then the number of permutations is

$$\frac{m!}{a_1! a_2! \cdots a_n!}$$

Multinomial Coefficients

$$\binom{m}{a_1, a_2, \ldots, a_n} = \frac{m!}{a_1! a_2! \cdots a_n!}$$

$$(x_1 + x_2 + \cdots + x_n)^m = \sum_{\substack{a_1 + a_2 + \cdots + a_n = m \\ a_1 \geq 0, \ldots, a_n \geq 0}} \binom{m}{a_1, a_2, \ldots, a_n} x_1^{a_1} x_2^{a_2} \ldots x_n^{a_n}.$$

E.g.

$$\begin{aligned} (x_1 + x_2 + x_3)^4 &= \binom{4}{3,0,0} x_1^4 + \binom{4}{3,1,0} x_1^3 x_2 + \\ &\quad \binom{4}{3,0,1} x_1^3 x_3 + \binom{4}{2,1,1} x_1^2 x_2 x_3 + \cdots \\ &= x_1^4 + 4x_1^3 x_2 + 4x_1^3 x_3 + 6x_1^2 x_2 x_3 + \cdots \end{aligned}$$

Contribution of 1 to the coefficient of $x_1^{a_1} x_2^{a_2} \ldots x_n^{a_n}$ from every permutation in $\{x_1 \times a_1, x_2 \times a_2, \ldots, x_n \times a_n\}$.

E.g.

$$(x_1 + x_2 + x_3)^6 = \cdots + x_2 x_3 x_2 x_1 x_1 x_3 + \cdots$$

where the displayed term comes by choosing x_2 from first bracket, x_3 from second bracket etc.

Balls in boxes

m distinguishable balls are placed in n distinguishable boxes. Box i gets b_i balls.

$$\# \text{ ways is } \binom{m}{b_1, b_2, \ldots, b_n}.$$

$m = 7, n = 3, b_1 = 2, b_2 = 2, b_3 = 3$
No. of ways is 7!/(2!2!3!)=210

$[1,2][3,4][5,6,7]$ $[1,2][3,5][4,6,7]$ $\cdots$ $[6,7][4,5][1,2,3]$

3 1 3 2 1 3 2

Ball 1 goes in box 3, Ball 2 goes in box 1, etc.