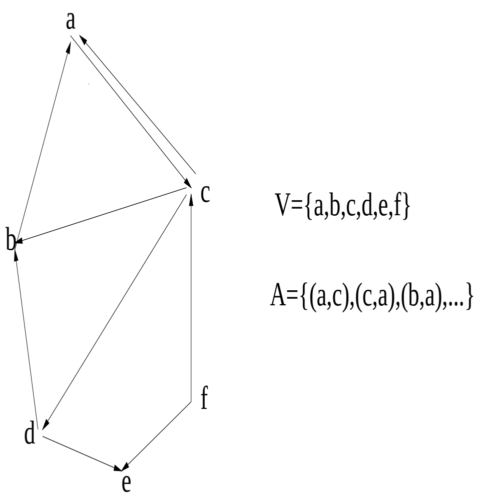
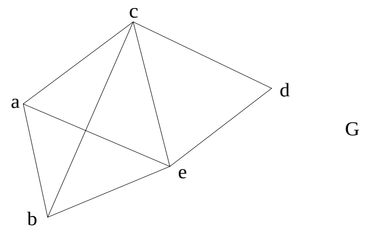
Digraphs

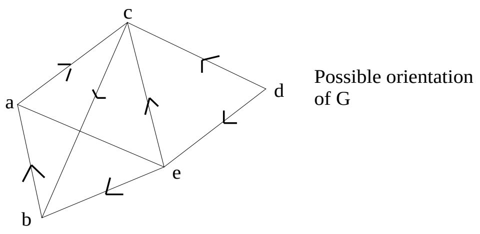
A *Digraph* $D = (V, A)$ has vertex set D and a set $A \subseteq V \times V$ of ordered pairs called *arcs*. [We do not allow (v, v) as an arc here.]



If we ignore the directions of the arcs in D, we get the underlying graph $G(D)$.

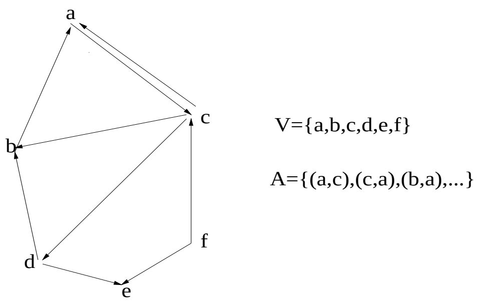
Also, given a graph $G = (V, E)$ we can *orient* its edges in $2^{|E|}$ ways to obtain a digraph.





Walks and Paths

A directed walk is a sequence $(v_1, v_2, \ldots, v_k)$ of vertices where $(v_i, v_{i+1}) \in A$ for $1 \leq i < k$. Thus



(a, c, d, b) is a directed walk.

(a, b, d, e) is not a directed walk.

A directed path is a directed walk which visits any vertex at most once.

A directed cycle is a closed directed path.

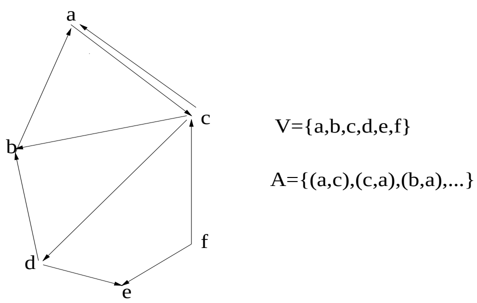
One can show as in the case of graphs that a directed walk of minimum length from a to b is always a directed path.

Strong Components

We define a relation $\sim$ on V by $a \sim b$ iff there is a directed walk (or path) from a to b and a directed walk from b to a.

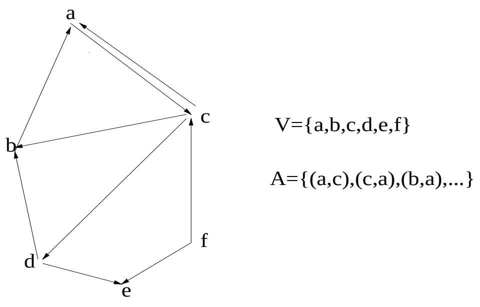
$\sim$ is an equivalence relation and its equivalence classes are called *strong components*.

A digraph is *strongly connected* if there is only one equivalence class i.e. there is a directed path from a to b for all $a, b \in V$.



Components: $\{a, b, c, d\}$, $\{e\}, \{f\}$.

Indegree $d^-(v)$ = number of arcs $(x, v), x \in V$.
Outdegree $d^+(v)$ = number of arcs $(v, x), x \in V$.



$d^+(a) = 1, d^-(a) = 2$ etc.

Lemma 1

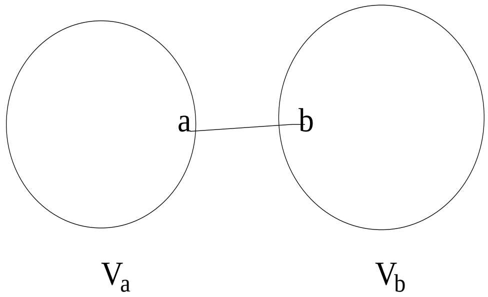
$$\sum_{v \in V} d^+(v) = \sum_{v \in V} d^-(v).$$

Strong Orientations

A *strong orientation* of a graph G is an orientation of its edges to make a digraph which is strongly connected.

Theorem 1 *A connected graph $G = (V, E)$ has a strong orientation iff G has no cut-edges.*

Proof Suppse G has a cut-edge $e = (a, b)$.

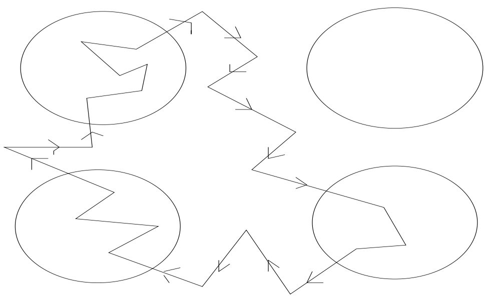


Let V_a, V_b be the components of $G - e$. If (a, b) is oriented from a to b (resp. from b to a) then there are no directed paths from V_b to V_a (resp. V_a to V_b).

only if: Let $e_1, \ldots, e_k$ be the edges G. G has no cut-edges so for every e_i there exists a cycle C_i which contains e_i.

Let $D_1 = (V(G), A_1)$ where A_1 is the set of edges of C_1 oriented in the same direction along the cycle.

Similarly, for $1 \leq i < k$, $D_{i+1} = (V(G), A_{i+1})$, where $A_{i+1} = A_i \cup B_{i+1}$ and B_{i+1} is a similar orientation – along C_{i+1} – of *unoriented* edges, $E(C_{i+1}) \setminus E(G(D_i))$.

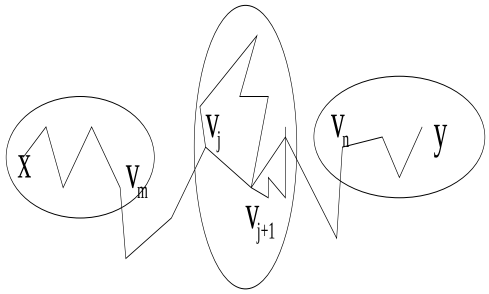


Let $K_i^1, \ldots, K_i^l$ be the *weak* components of D_i, i.e. the connected components if we ignore the directions of the arcs in D_i. We show, that for every i, all *weak* components K_i^j are strong ones, too. When $i = k$ then there is only one weak component – G.

We proceed by induction on i. In case $i = 1$, D_1 is single a cycle where all edges are oriented in the same direction: it is clearly strongly connected.

Take two arbitrary vertices x and y of $K_{i+1}^{j} \subseteq D_{i+1}$. If both of them are in the same weak component of D_i then – by induction – there are oriented paths from x to y and from y to x. So assume that x and y are in two different *strong* components of D_i.

Let $C_{i+1} = (v_1, v_2, \dots, v_l, v_1)$ and let $K_i^{v_j}$ be the *strong* component in D_i containing the vertex $v_j \in C_{i+1}$. Since x and y are in the same *weak* component of D_{i+1}, $x \in K_i^{v_m}$ and $y \in K_i^{v_n}$ for some $v_m, v_n \in C_{i+1}$. To reach y from x do the following. First go from x to v_m on a directed path. This can be done, since both of them are in the same strong component $K_i^{v_m}$.

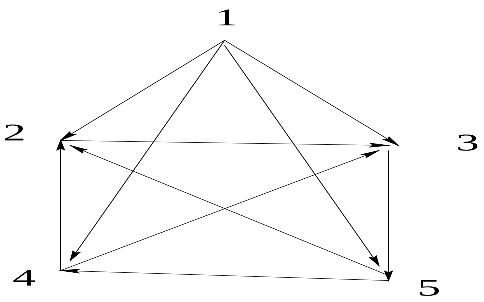


Then go along a directed path from v_m to v_n. This also can be done, because if there is no directed edge between two consecutive vertices v_j and v_{j+1} along a cycle, then both of them are in the same component of D_i and by induction, there exist a directed path from v_j to v_{j+1}. Finally, go from v_n to y. Again, this can be done, since both of them are in the same strong component $K_i^{v_n}$.

Similarly, x too can be reached from y via an oriented path.

Tournaments

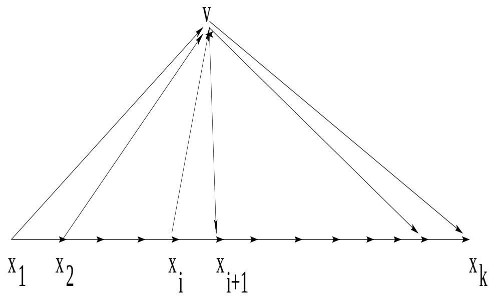
A tournament is an orientation of a complete graph K_n.



A directed Hamilton Path is a directed path which visits every vertex. Thus (1,2,3,5,4) is a Hamilton path in the tournament above.

Theorem 2 *A tournament contains a directed Hamilton path.*

Proof Let D be a tournament and let $P = (x_1, x_2, \dots, x_k)$ be a directed path of maximum length in D. We claim that P is a directed Hamilton path.

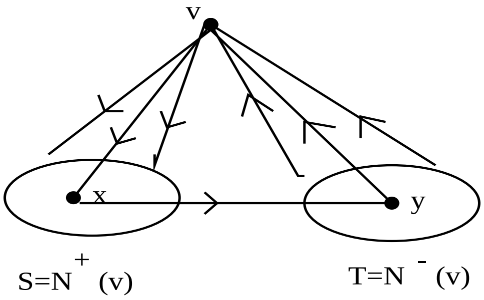


Suppose $\exists v \notin \{x_1, x_2, \dots, x_k\}$. The edge (x_1, v) must be oriented from x_1 to v or (v, P) is a directed path which is longer than P. Similarly, (v, X_k) must be oriented from v to x_k.

So, there must be $1 \leq i < k$ such that (x_i, v) is oriented from x_i to v and (v, x_{i+1}) is oriented from v to x_{i+1}. But then D contains the path $x_1, \dots, x_i, v, x_{i+1}, \dots, x_k)$ which is longer than P – contradiction.

Theorem 3 *If D is a strongly connected tournament with $n \geq 3$ vertices then D contains a directed cycle of size k for all $3 \leq k \leq n$.*

Proof By induction on k.
$k = 3$.
Choose $v \in V$ and let $S = N^+(V)$, $T = N^-(v) = V \setminus (S \cup \{v\})$.

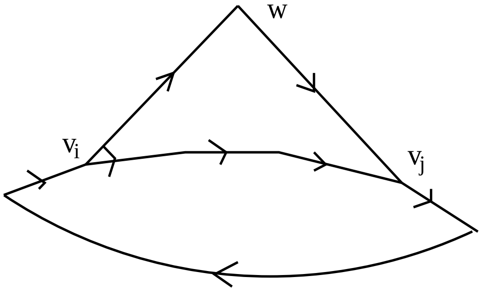


$S \neq \emptyset$ since D is strongly connected. Similarly, $S \neq V \setminus \{v\}$ else $N^+(V \setminus \{v\}) = \emptyset$.

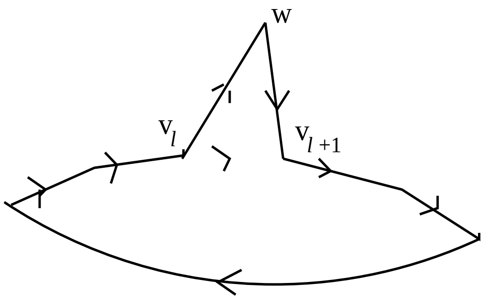
Thus $N^+(S) \neq \emptyset$. $v \notin N^+(S)$ and so $N^+(S) = T$.
Thus $\exists x \in S, y \in T$ with $xy \in A$.

Suppose now that there exists a directed cycle $C = (v_1, v_2, \ldots, v_k, v_1)$.

Case 1: $\exists w \notin C$ and $i \neq j$ such that $v_i w \in A$, $w v_j \in A$.

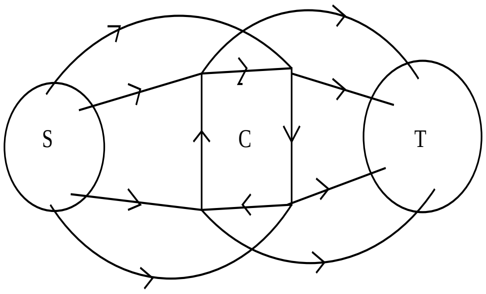


It follows that there exists ℓ with $v_\ell w \in A$, $w v_{\ell+1} \in A$.
$C' = (w, v_{\ell+1}, \ldots, v_\ell, v_1, \ldots, v_\ell, w)$ is a cycle of length $k+1$.



Case 2 $V \setminus C = S \cup T$ where

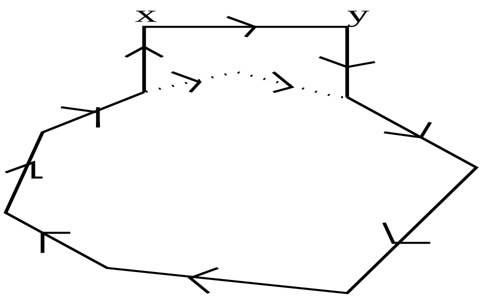
$w \in S$ implies $w v_i \in A$, $1 \leq i \leq k$.
$w \in T$ implies $v_i w \in A$, $1 \leq i \leq k$.



$S = \emptyset$ implies $T = \emptyset$ (and C is a Hamilton cycle) or $N^+(T) = \emptyset$.
$T = \emptyset$ implies $N^+(C) = \emptyset$.

Thus we can assume
$S, T \neq \emptyset$ and $N^+(T) \neq \emptyset$.
$N^+(T) \cap C = \emptyset$ and so $N^+(T) \cap S \neq \emptyset$.

Thus $\exists x \in T, y \in S$ such that $xy \in A$.



The cycle $(v_1, x, y, v_3, \ldots, v_k, v_1)$ is a cycle of length $k + 1$.