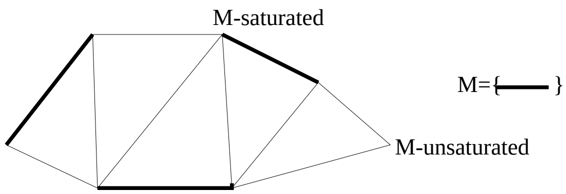
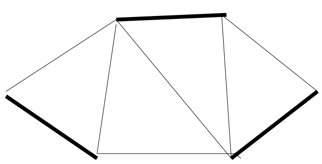
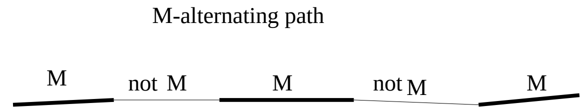
Matchings

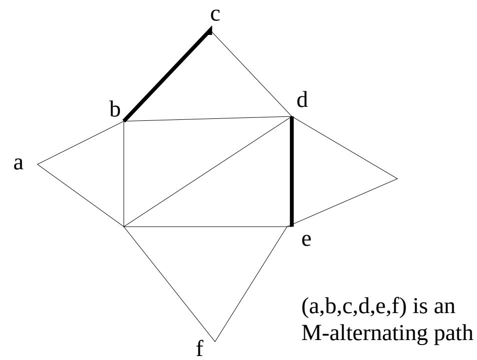
A *matching* M of a graph $G = (V, E)$ is a set of edges, no two of which are incident to a common vertex.





Perfect Matching



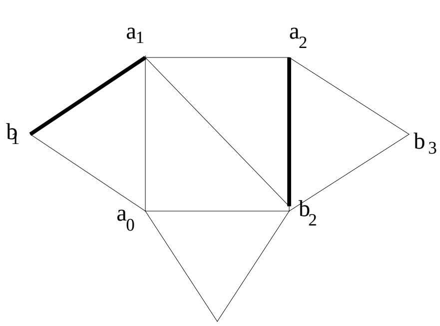


An M-alternating path joining 2 M-unsaturated vertices is called an M-augmenting path.

M is a *maximum* matching of G if no matching M' has more edges.

Theorem 1 *M is a maximum matching iff M admits no M-augmenting paths.*

Proof Suppose M has an augmenting path $P = (a_0, b_1, a_1, \ldots, a_k, b_{k+1})$ where $e_i = (a_{i-1}, b_i) \notin M$, $1 \leq i \leq k+1$ and $f_i = (b_i, a_i) \in M$, $1 \leq i \leq k$.



$$M' = M - \{f_1, f_2, \ldots, f_k\} + \{e_1, e_2, \ldots, e_{k+1}\}.$$

- $|M'| = |M| + 1$.
- M' is a matching

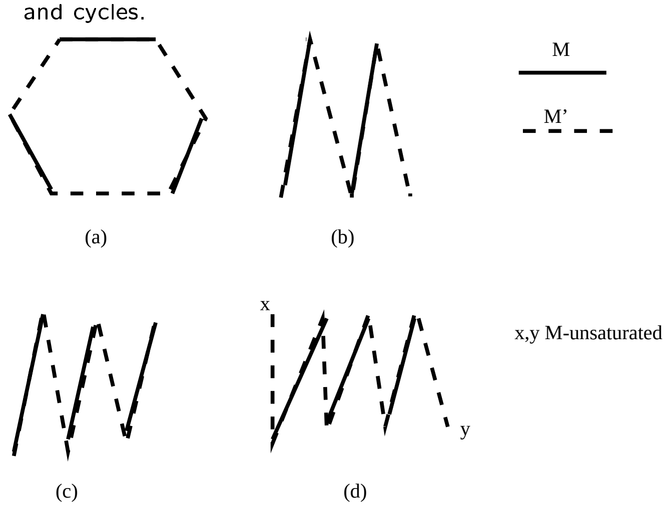
For $x \in V$ let $d_M(x)$ denote the degree of x in matching M, So $d_M(x)$ is 0 or 1.

$$d_{M'}(x) = \begin{cases} d_M(x) & x \notin \{a_0, b_1, \ldots, b_{k+1}\} \\ d_M(x) & x \in \{b_1, \ldots, a_k\} \\ d_M(x) + 1 & x \in \{a_0, b_{k+1}\} \end{cases}$$

So if M has an augmenting path it is not maximum.

Suppose M is not a maximum matching and $|M'| > |M|$. Consider $H = G[M \nabla M']$ where $M \nabla M' = (M \setminus M') \cup (M' \setminus M)$ is the set of edges in *exactly* one of M, M'.

Maximum degree of H is 2 – ≤ 1 edge from M or M'. So H is a collection of vertex disjoint alternating paths and cycles.



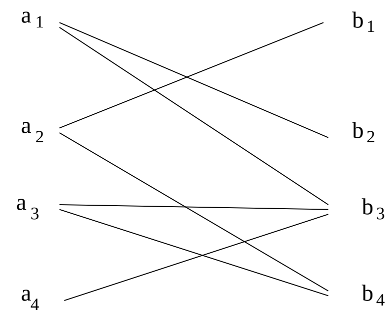
$|M'| > |M|$ impplies that there is at least one path of type (d).

Such a path is M-augmenting □

Bipartite Graphs

Let $G = (A \cup B, E)$ be a bipartite graph with bipartition A, B.

For $S \subseteq A$ let $N(S) = \{b \in B : \exists a \in S, (a,b) \in E\}$.



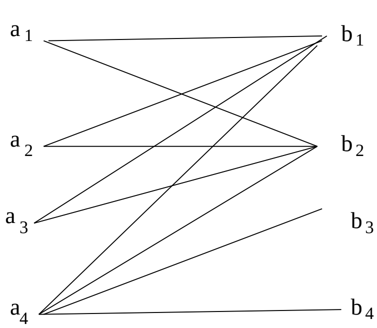
N({a$_2$, a$_3$ }) ={ b$_1$b$_3$ b$_4$ }

Clearly, $|M| \leq |A|, |B|$ for any matching M of G.

Hall's Theorem

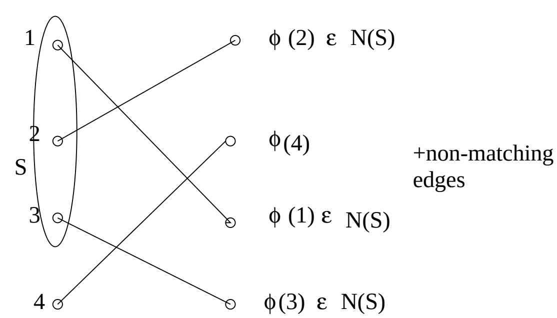
Theorem 2 *G contains a matching of size $|A|$ iff*

$$|N(S)| \geq |S| \qquad \forall S \subseteq A. \qquad (1)$$



$N(\{a_1, a_2, a_3\}) = \{b_1, b_2\}$ and so at most 2 of a_1, a_2, a_3 can be saturated by a matching.

Only if: Suppose $M = \{(a, \phi(a)) : a \in A\}$ saturates A.

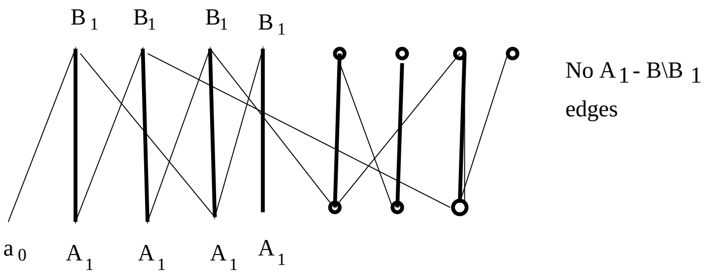


$$\begin{aligned} |N(S)| &\geq |\{\phi(s) : s \in S\}| \\ &= |S| \end{aligned}$$

and so (1) holds.

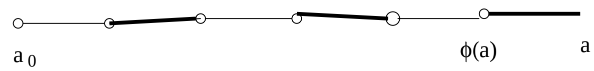
If: Let $M = \{(a, \phi(a)) : a \in A'\}$ $(A' \subseteq A)$ is a maximum matching. Suppose $a_0 \in A$ is M-unsaturated. We show that (1) fails.

Let
$A_1 = \{a \in A :$ such that a is reachable from a_0 by an M-alternating path.$\}$
$B_1 = \{b \in B :$ such that b is reachable from a_0 by an M-alternating path.$\}$

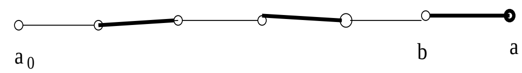


- B_1 is M-saturated else there exists an M-augmenting path.

- If $a \in A_1 \setminus \{a_0\}$ then $\phi(a) \in B_1$.



- If $b \in B_1$ then $\phi^{-1}(b) \in A_1 \setminus \{a_0\}$.



So

$$|B_1| = |A_1| - 1.$$

- $N(A_1) \subseteq B_1$

So

$$|N(A_1)| = |A_1| - 1$$

and (1) fails to hold.

Marriage Theorem

Theorem 3 *Suppose $G = (A \cup B, E)$ is k-regular. ($k \geq 1$) i.e. $d_G(v) = k$ for all $v \in A \cup B$. Then G has a perfect matching.*

Proof

$$k|A| = |E| = k|B|$$

and so $|A| = |B|$.

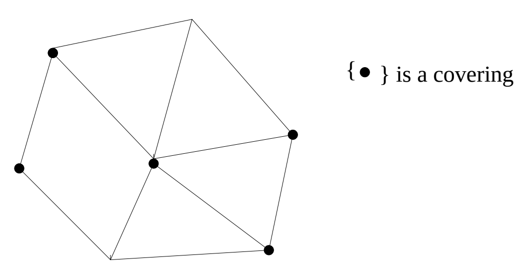
Suppose $S \subseteq A$. Let m be the number of edges incident with S. Then

$$k|S| = m \leq k|N(S)|.$$

So (1) holds and there is a matching of size $|A|$ i.e. a perfect matching.

Edge Covers

A set of vertices $X \subseteq V$ is a *covering* of $G = (V, E)$ if every edge of E contains at least one endpoint in X.



Lemma 1 *If X is a covering and M is a matching then $|X| \leq |M|$.*

Proof Let $M = \{(a_1, b_i) : 1 \leq i \leq k\}$. Then $|X| \geq |M|$ since $a_i \in X$ or $b_i \in X$ for $1 \leq i \leq k$ and $a_1, \ldots, b_k$ are distinct. □

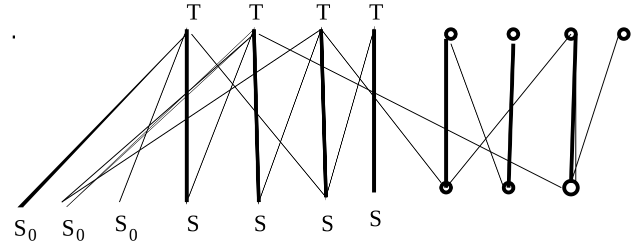
Konig's Theorem

Let $\mu(G)$ be the maximum size of a matching.
Let $\beta(G)$ be the minimum size of a covering.
Then

$$\mu(G) \leq \beta(G).$$

Theorem 4 *If G is* bipartite *then $\mu(G) = \beta(G)$.*

Proof Let M be a maximum matching.
Let S_0 be the M-unsaturated vertices of A.
Let $S \supseteq S_0$ be the A-vertices which are reachable from S by M-alternating paths.
Let T be the M-neighbours of $S \setminus S_0$.



Let $X = (A \setminus S) \cup T$.
• $|X| = |M|$.
$|T| = |S \setminus S_0|$. The remaining edges of M cover $A \setminus S$ exactly once.

• X is a cover.
There are no edges (x, y) where $x \in S$ and $y \in B \setminus T$. Otherwise, since y is M-saturated (no M-augmenting paths) the M-neightbour of y would have to be in S, contradicting $y \notin T$. □