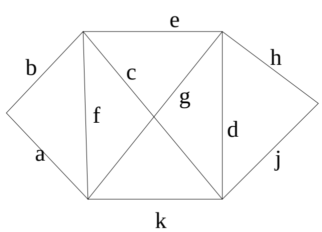
Eulerian Graphs

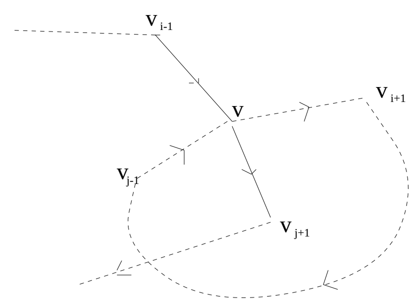
An *Eulerian cycle* of a graph $G = (V, E)$ ia closed walk which uses each edge $e \in E$ exactly once.



The walk using edges a,b,c,d,e,f,g,h,j,k in this order is an Eulerian cycle.

Theorem 1 *A connected graph is Eulerian i.e. has an Eulerian cycle, iff it has no vertex of odd degree.*

Proof Suppose $W = (v_1, v_2, \ldots, v_m, v_1)$ ($m = |E|$) is an Eulerian cycle. Fix $v \in V$. Whenever W visits v it enters through a new edge and leaves through a new edge. Thus each visit requires 2 new edges. Thus the degree of v is even.



The converse is proved by induction on $|E|$. The result is true for $|E| = 3$. The only possible graph is a triangle.

Assume $|E| \geq 4$. G is not a tree, since it has no vertex of degree 1. Therefore it contains a cycle C. Delte the edges of C. The remaining graph has components $K_1, K_2, \ldots, K_r$.

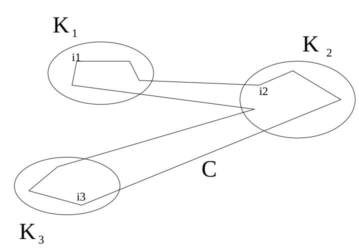
Each K_i is connected and is of even degree – deleting C removes 0 or 2 edges incident with a given $v \in V$. Also, each K_i has strictly less than $|E|$ edges. So, by induction, each JK_i has an Eulerain cycle, C_i say.

We create an Eulerian cycle of G as follows: let $C = (v_1, v_2, \ldots, v_s, v_1)$. Let v_{i_t} be the first vertex of C which is in K_t. Assume w.l.o.g. that $i_1 < i_2 < \cdots < i_r$.

$$W = (v_1, v_2, \ldots, v_{i_1}, C_1, , v_{i_1}, \ldots, v_{i_2}, C_2, v_{i_2}, \ldots, v_{i_r}, C_r, v_{i_r}, \ldots, v_1)$$

is an Eulerian cycle of G. □

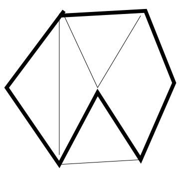
x



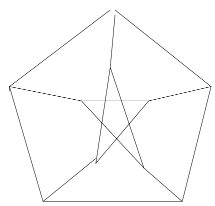
Hamilton Cycles

A *Hamilton Cycle* of a graph $G = (V, E)$ is a cycle which goes through each vertex (once).

A graph is called *Hamiltonian* if it contains a Hamilton cycle.



Hamiltonian Graph



Non-Hamiltonian Graph
Petersen Graph

Lemma 1 *Let $G = (V, E)$ and $|V| = n$. Suppose $x, y \in V$, $e = (x, y) \notin E$ and $d(x) + d(y) \geq n$. Then*

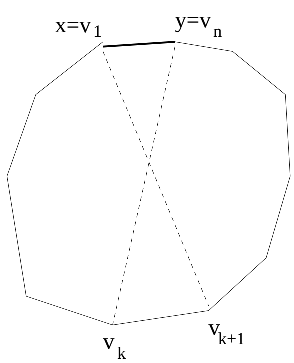
$$G + e \text{ is Hamiltonian } \leftrightarrow G \text{ is Hamiltonian.}$$

Proof
$\leftarrow$ Trivial.

$\rightarrow$ Suppose $G + e$ has a Hamilton cycle H. If $e \notin H$ then $H \subseteq G$ and G is Hamiltonian.

Suppose $e \in H$. We show that we can find another Hamilton cycle in $G + e$ which does not use e.

x



$H = (x = v_1, v_2, \ldots, v_n = y, x)$.
$S = \{i : (x, v_{i+1}) \in E\}$, $T = \{i : (y, v_i) \in E\}$.

$S \subseteq \{1, 2, \ldots, n-2\}$, $T \subseteq \{2, 3, \ldots, n-1\}$.
$|S| + |T| \geq n$ and $|S \cup T| \leq n - 1$.

Thus

$$|S \cap T| = |S| + |T| - |S \cup T| \geq 1$$

and so $\exists 1 \neq k \in S \cap T$ and then

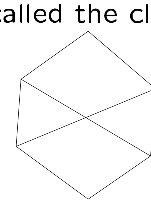
$$H' = (v_1, v_2, \ldots, v_k, v_n, v_{n-1}, \ldots, v_{k+1}, v_1)$$

is a Hamilton cycle of G.

Bondy-Chvatál Closure of a graph

begin
  $\widehat{G} := G$
  while $\exists (x, y) \notin E$ with $d_{\widehat{G}}(x) + d_{\widehat{G}}(y) \geq n$ **do**
  begin
    $\widehat{G} := \widehat{G} + (x, y)$
  end
  Output $\widehat{G}$
end

The graph $\widehat{G}$ is called the closure of G.



Lemma 2 *$\widehat{G}$ is independent of the order in which edges are added i.e. it depends only on G.*

Proof Suppose algorithm is run twice to obtain
$G_1 = G + e_1 + e_2 + \cdots + e_k$ and
$G_2 = G + f_1 + f_2 + \cdots + f_\ell$.
We show that $\{e_1, e_2, \ldots, e_k\} = \{f_1, f_2, \ldots, f_\ell\}$.

Suppose not. Let $t = \min\{i : e_i \notin G_2\}$, $e_t = (x, y)$ and $G' = G + e_1 + e_2 + \cdots + e_{t-1}$. Then

$$\begin{aligned} d_{G_2}(x) + d_{G_2}(y) &\geq d_{G'}(x) + d_{G'}(y) \\ &\geq n \end{aligned}$$

since e_t was added to G'.

But then e_t should have been added to G_2 – contradiction.

- $\widehat{G}$ Hamiltonian $\Rightarrow$ G is Hamiltonian.

- $\widehat{G}$ complete $\Rightarrow$ G is Hamiltonian.

- $\delta(G) \geq n/2 \Rightarrow G$ is Hamiltonian.

Second statement is due to Bondy and Murty. Third statement is due to Dirac.