Properties of binomial coefficients

- Symmetry

$$\binom{n}{r} = \binom{n}{n-r}$$

Choosing r elements to include is equivalent to choosing $n-r$ elements to exclude.

- Pascal's Triangle

$$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$$

A $k+1$-subset of $[n+1]$ either (i) includes $n+1$ – $\binom{n}{k}$ choices or (ii) does not include $n+1$ – $\binom{n}{k+1}$ choices.

Generalisation

$$\binom{k}{k} + \binom{k+1}{k} + \binom{k+2}{k} + \cdots + \binom{n}{k} = \binom{n+1}{k+1}. \quad (1)$$

Proof 1: induction on n for arbitrary k.

Base case: $n = k$;

$\binom{k}{k} = \binom{k+1}{k+1}$

Inductive Step: assume true for $n \geq k$.

$$\begin{aligned} \sum_{m=k}^{n+1} \binom{m}{k} &= \sum_{m=k}^{n} \binom{m}{k} + \binom{n+1}{k} \\ &= \binom{n+1}{k+1} + \binom{n+1}{k} \quad \text{Induction} \\ &= \binom{n+2}{k+1}. \quad \text{Pascal's triangle} \end{aligned}$$

Proof 2: Combinatorial argument.

If S denotes the set of $k+1$-subsets of $[n+1]$ and S_m is the set of $k+1$-subsets of $[n+1]$ which have largest element $m+1$ then

- $S_k, S_{k+1}, \ldots, S_n$ is a partition of S.
- $|S_k| + |S_{k+1}| + \cdots + |S_n| = |S|$.
- $|S_m| = \binom{m}{k}$.

This proves the result.

Vandermonde's Identity

$$\sum_{r=0}^{k} \binom{m}{r}\binom{n}{k-r} = \binom{m+n}{k}. \quad (2)$$

Ex:

$$\binom{6}{0}\binom{8}{4}+\binom{6}{1}\binom{8}{3}+\binom{6}{2}\binom{8}{2}+\binom{6}{3}\binom{8}{1}+\binom{6}{4}\binom{8}{0} = \binom{14}{4}$$

Split $[m+n]$ into $A = [m]$ and $B = [m+n]\setminus[m]$. Let S denote the set of k-subsets of $[m+n]$ and let $S_r = \{X \in S : |X \cap A| = r\}$. Then

- $S_0, S_1, \ldots, S_m$ is a partition of S.
- $|S_0| + |S_1| + \cdots + |S_m| = |S|$.
- $|S_r| = \binom{m}{r}\binom{n}{k-r}$.
- $|S| = \binom{m+n}{k}$.

This proves the identity.

Binomial Theorem

$$(1+x)^n = \sum_{r=0}^{n} \binom{n}{r} x^r. \qquad (3)$$

Coefficient x^r in $(1+x)(1+x)\cdots(1+x)$: choose x from r brackets and 1 from the rest.

The proof of equation (3) assumed that n was an integer. The binomial theorem remains true for all real (or complex) n provided $|x| \le 1$ i.e.

$$(1+x)^\alpha = \sum_{r=0}^{\infty} \binom{\alpha}{r} x^r$$

where $\binom{\alpha}{r} = \alpha(\alpha-1)\cdots(\alpha-r+1)/r!$ – proof in any standard calculus text book.

Newton's Binomial Theorem

$$(1+x)^\alpha = \sum_{k=0}^{\infty} \frac{\alpha(\alpha-1)\ldots(\alpha-k+1)}{k!} x^k,$$

for real α and $|x| < 1$.

$f(x) = (1+x)^\alpha$
$f^{(k)}(x) = \alpha(\alpha-1)\ldots(\alpha-k+1)(1+x)^{\alpha-k}$.
$f^{(k)}(0) = \alpha(\alpha-1)\ldots\alpha-k+1$.
Taylor's theorem

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$$

yields theorem

$$\begin{aligned}(1-x)^{-m} &= \sum_{k=0}^{\infty} \frac{(-m)(-m-1)\ldots(-m-k+1)}{k!}(-x)^k \\ &= \sum_{k=0}^{\infty} \frac{m(m+1)\ldots(m+k-1)}{k!} x^k \\ &= \sum_{k=0}^{\infty} \binom{m+k-1}{k} x^k.\end{aligned}$$

So if $m = 3$ then

$$\begin{aligned}\frac{1}{(1-x)^3} &= \binom{2}{0} + \binom{3}{1}x + \binom{4}{2}x^2 + \cdots + \binom{n+2}{n}x^n + \cdots \\ &= 1 + 3x + 6x^2 + \cdots + \frac{n(n+1)}{2}x^n + \cdots\end{aligned}$$

$$\begin{aligned}(1+x)^{1/2} &= 1 + \sum_{k=1}^{\infty} \frac{(1/2)(1/2-1)\ldots(1/2-k+1)}{k!} x^k \\ &= 1 + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{2^k} \frac{1 \times 3 \times \cdots \times (2k-3)}{k!} x^k \\ &= 1 + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{2^k} \frac{(2k-2)!}{(2 \times 4 \times \cdots \times (2k-2))k!} x^k \\ &= 1 + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{2^k} \frac{(2k-2)!}{2^{k-1}(k-1)!k!} x^k \\ &= 1 + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k2^{2k-1}} \binom{2k-2}{k-1} x^k\end{aligned}$$

Applications

- $x = 1$:

  $$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = (1+1)^n = 2^n.$$

  LHS counts the number of subsets of all sizes in $[n]$.

- $x = -1$:

  $$\binom{n}{0} - \binom{n}{1} + \cdots + (-1)^n\binom{n}{n} = (1-1)^n = 0,$$

  i.e.

  $$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots$$

  and number of subsets of even cardinality = number of subsets of odd cardinality.

$$\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1}.$$

Differentiate both sides of the Binomial Theorem w.r.t. x.

$$n(1+x)^{n-1} = \sum_{k=0}^{n} k\binom{n}{k}x^{k-1}.$$

Now put $x = 1$.

Pascal's Triangle

The following array of binomial coefficents, constitutes the famous triangle:

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
...