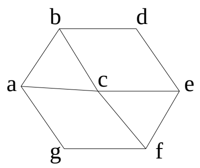
Paths and Walks

$W = (v_1, v_2, \ldots, v_k)$ is a walk in G if $(v_i, v_{i+1}) \in E$ for $1 \leq i < k$.

A path is a walk in which the vertices are distinct.

W_1 is a path, but W_2, W_3 are not.

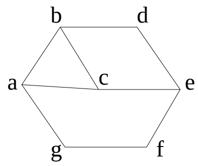


W_1 = a,b,c,e,d
W_2 =a,b,a,c,e
W_3 =g,f,c,e,f

A walk is *closed* if $v_1 = v_k$. A *cycle* is a closed walk in which the vertices are distinct except for v_1, v_k.

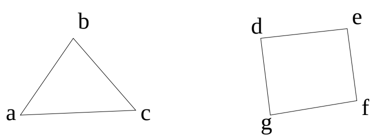
b, c, e, d, b is a cycle.

b, c, a, b, d, e, c, b is not a cycle.



Connected components

We define a relation $\sim$ on V.
$a \sim b$ iff there is a walk from a to b.



$a \sim b$ but $a \not\sim d$.

Claim: $\sim$ is an equivalence relation.

reflexivity $v \sim v$ as v is a (trivial) walk from v to v.

Symmetry $u \sim v$ implies $v \sim u$.
$(u = u_1, u_2 \ldots, u_k = v)$ is a walk from u to v implies $(u_k, u_{k-1}, \ldots, u_1)$ is a walk from v to u.

Transitivity $u \sim v$ and $v \sim w$ implies $u \sim w$.
$W_1 = (u = u_1, u_2 \ldots, u_k = v)$ is a walk from u to v and $W_2 = (v_1 = v, v_2, v_3, \ldots, v_\ell = w)$ is a walk from v to w imples that $(W_1, W_2) = (u_1, u_2 \ldots, u_k, v_2, v_3, \ldots, v_\ell)$ is a walk from u to w.

The equivalence classes of $\sim$ are called *connected components*.

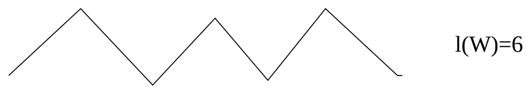
In general $V = C_1 \cup V_2 \cup \cdots \cup C_r$ where $C_1, C_2, \ldots, C_r$ are the connected comonents.

We let $\omega(G)(= r)$ be the number of components of G.

G is *connected* iff $\omega(G) = 1$ i.e. there is a walk between every pair of vertices.

Thus $C_1, C_2, \ldots, C_r$ induce connected subgraphs $G[C_1], \ldots, G[C_r]$ of G

For a walk W we let $\ell(W)$ = no. of edges in W.



Lemma 1 *Suppose W is a walk from vertex a to vertex b and that W minimises ℓ over all walks from a to b. Then W is a path.*

Proof Suppose $W = (a = a_0, a_1, \ldots, a_k = b)$ and $a_i = a_j$ where $0 \leq i < j \leq k$. Then $W' = (a_0, a_1, \ldots, a_i, a_{j+1}, \ldots, a_k)$ is also a walk from a to b and $\ell(W') = \ell(W) - (j-i) < \ell(W)$ – contradiction. $\square$

Corollary 1 *If $a \sim b$ then there is a path from a to b.*

So G is connected $\leftrightarrow$ $\forall a, b \in V$ there is a path from a to b.

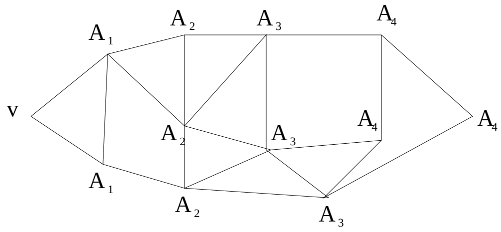
Breadth First Search – BFS

Fix $v \in V$. For $w \in V$ let

$d(v, w)$ = length of shortest path from v to w.

For $t = 0, 1, 2, \ldots,$ let

$$A_t = \{w \in V : \ d(v, w) = t\}.$$



$A_0 = \{v\}$ and $v \sim w \leftrightarrow d(v, w) < \infty$.

In BFS we construct $A_0, A_1, A_2, \ldots,$ by

$$A_{t+1} = \{w \notin A_0 \cup A_1 \cup \cdots \cup A_t : \ \exists \text{ an edge } (u, w) \text{ such that } u \in A_t\}.$$

Note : no edges (a, b) between A_k and A_ℓ for $\ell - k \geq 2$, else $w \in A_{k+1} \neq A_\ell$. (1)

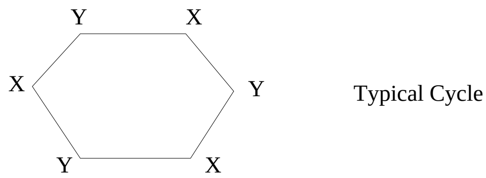
In this way we can find all vertices in the same component C as v.

By repeating for $v' \notin C$ we find another component etc.

Characterisation of bipartite graphs

Theorem 1 *G is bipartite $\leftrightarrow$ G has no cycles of odd length.*

Proof $\rightarrow$: $G = (X \cup Y, E)$.



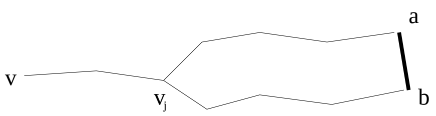
Suppose $C = (u_1, u_2, \ldots, u_k, u_1)$ is a cycle. Suppose $u_1 \in X$. Then $u_2 \in Y, u_3 \in X, \ldots, u_k \in Y$ implies k is even.

$\leftarrow$ Assume G is connected, else apply following argument to each component.
Choose $v \in V$ and construct $A_0, A_1, A_2, \ldots,$ by BFS.

$X = A_0 \cup A_2 \cup A_4 \cup \cdots$ and $Y = A_1 \cup A_3 \cup A_5 \cup \cdots$

We need only show that X and Y contain no edges and then all edges must join X and Y. Suppose X contains edge (a, b) where $a \in A_k$ and $b \in A_\ell$.

(i) If $k \neq \ell$ then $|k - \ell| \geq 2$ which contradicts (1)
(ii) $k = \ell$:



There exist paths $(v = v_0, v_1, v_2, \ldots, v_k = a)$ and $(v = w_0, w_1, w_2, \ldots, w_k = b)$.

Let $j = \max\{t : v_t = w_t\}$.

$$(v_j, v_{j+1}, \ldots, v_k, w_k, w_{k-1}, \ldots, w_j)$$

is an odd cycle – length $2(k - j) + 1$ – contradiction. $\square$