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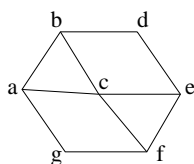
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Paths and Walks

$W = (v_1, v_2, \dots, v_k)$ is a walk in G if $(v_i, v_{i+1}) \in E$ for $1 \leq i < k$.

A path is a walk in which the vertices are distinct.

W_1 is a path, but W_2, W_3 are not.



$W_1 = a, b, c, e, d$

$W_2 = a, b, a, c, e$

$W_3 = g, f, c, e, f$

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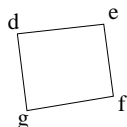
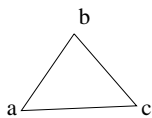
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Connected components

We define a relation \sim on V .

$a \sim b$ iff there is a walk from a to b .



$a \sim b$ but $a \not\sim d$.

Claim: \sim is an equivalence relation.

reflexivity $v \sim v$ as v is a (trivial) walk from v to v .

Symmetry $u \sim v$ implies $v \sim u$.

$(u = u_1, u_2, \dots, u_k = v)$ is a walk from u to v implies $(u_k, u_{k-1}, \dots, u_1)$ is a walk from v to u .

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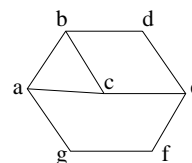
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A walk is *closed* if $v_1 = v_k$. A *cycle* is a closed walk in which the vertices are distinct except for v_1, v_k .

b, c, e, d, b is a cycle.

b, c, a, b, d, e, c, b is not a cycle.



Transitivity $u \sim v$ and $v \sim w$ implies $u \sim w$.

$W_1 = (u = u_1, u_2, \dots, u_k = v)$ is a walk from u to v and $W_2 = (v_1 = v, v_2, v_3, \dots, v_\ell = w)$ is a walk from v to w implies that $(W_1, W_2) = (u_1, u_2, \dots, u_k, v_2, v_3, \dots, v_\ell)$ is a walk from u to w .

The equivalence classes of \sim are called *connected components*.

In general $V = C_1 \cup V_2 \cup \dots \cup C_r$ where C_1, C_2, \dots, C_r are the connected components.

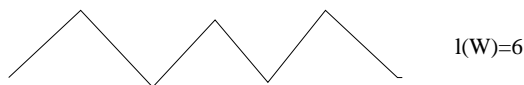
We let $\omega(G)(=r)$ be the number of components of G .

G is *connected* iff $\omega(G) = 1$ i.e. there is a walk between every pair of vertices.

Thus C_1, C_2, \dots, C_r induce connected subgraphs $G[C_1], \dots, G[C_r]$ of G

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For a walk W we let $\ell(W) = \text{no. of edges in } W$.



Lemma 1 Suppose W is a walk from vertex a to vertex b and that W minimises ℓ over all walks from a to b . Then W is a path.

Proof Suppose $W = (a = a_0, a_1, \dots, a_k = b)$ and $a_i = a_j$ where $0 \leq i < j \leq k$. Then $W' = (a_0, a_1, \dots, a_i, a_{j+1}, \dots, a_k)$ is also a walk from a to b and $\ell(W') = \ell(W) - (j - i) < \ell(W)$ – contradiction. \square

Corollary 1 If $a \sim b$ then there is a path from a to b .

So G is connected $\leftrightarrow \forall a, b \in V$ there is a path from a to b .

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In BFS we construct A_0, A_1, A_2, \dots , by

$$A_{t+1} = \{w \notin A_0 \cup A_1 \cup \dots \cup A_t : \exists \text{ an edge } (u, w) \text{ such that } u \in A_t\}.$$

Note : no edges (a, b) between A_k and A_ℓ for $\ell - k \geq 2$, else $w \in A_{k+1} \neq A_\ell$.
(1)

In this way we can find all vertices in the same component C as v .

By repeating for $v' \notin C$ we find another component etc.

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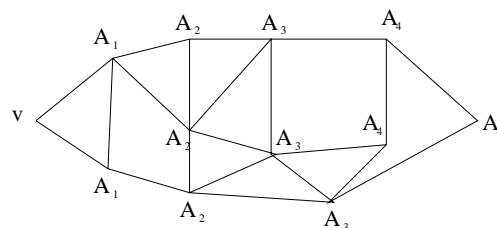
Breadth First Search – BFS

Fix $v \in V$. For $w \in V$ let

$d(v, w) = \text{length of shortest path from } v \text{ to } w$.

For $t = 0, 1, 2, \dots$, let

$$A_t = \{w \in V : d(v, w) = t\}.$$



$$A_0 = \{v\} \text{ and } v \sim w \leftrightarrow d(v, w) < \infty.$$

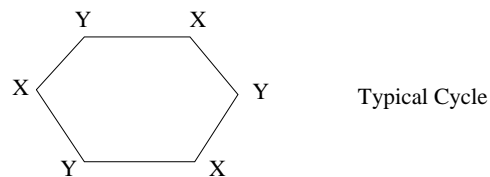
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Characterisation of bipartite graphs

Theorem 1 G is bipartite $\leftrightarrow G$ has no cycles of odd length.

Proof \rightarrow : $G = (X \cup Y, E)$.



Suppose $C = (u_1, u_2, \dots, u_k, u_1)$ is a cycle. Suppose $u_1 \in X$. Then $u_2 \in Y, u_3 \in X, \dots, u_k \in Y$ implies k is even.

\leftarrow Assume G is connected, else apply following argument to each component.

Choose $v \in V$ and construct A_0, A_1, A_2, \dots , by BFS.

$$X = A_0 \cup A_2 \cup A_4 \cup \dots \text{ and } Y = A_1 \cup A_3 \cup A_5 \cup \dots$$

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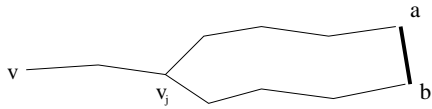
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We need only show that X and Y contain no edges and then all edges must join X and Y . Suppose X contains edge (a, b) where $a \in A_k$ and $b \in A_\ell$.

(i) If $k \neq \ell$ then $|k - \ell| \geq 2$ which contradicts (1)

(ii) $k = \ell$:



There exist paths $(v = v_0, v_1, v_2, \dots, v_k = a)$ and $(v = w_0, w_1, w_2, \dots, w_k = b)$.

Let $j = \max\{t : v_t = w_t\}$.

$$(v_j, v_{j+1}, \dots, v_k, w_k, w_{k-1}, \dots, w_j)$$

is an odd cycle – length $2(k - j) + 1$ – contradiction. \square

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