

Inclusion-Exclusion

2 sets:

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$$

So if $A_1, A_2 \subseteq A$ and $\bar{A}_i = A \setminus A_i$, $i = 1, 2$ then

$$|\bar{A}_1 \cap \bar{A}_2| = |A| - |A_1| - |A_2| + |A_1 \cap A_2|$$

3 sets:

$$\begin{aligned} |\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3| &= |A| - |A_1| - |A_2| - |A_3| \\ &\quad + |A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3| \\ &\quad - |A_1 \cap A_2 \cap A_3|. \end{aligned}$$

Simple example. How many integers in $[1000]$ are not divisible by 5,6 or 8 i.e. what is the size of $\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3$ below?

$A = A_\emptyset = \{1, 2, 3, \dots\}$	$ A = 1000$
$A_1 = \{5, 10, 15, \dots\}$	$ A_1 = 200$
$A_2 = \{6, 12, 18, \dots\}$	$ A_2 = 166$
$A_3 = \{8, 16, 24, \dots\}$	$ A_2 = 125$
$A_{\{1,2\}} = \{30, 60, 90, \dots\}$	$ A_{\{1,2\}} = 33$
$A_{\{1,3\}} = \{40, 80, 120, \dots\}$	$ A_{\{1,3\}} = 25$
$A_{\{2,3\}} = \{24, 48, 72, \dots\}$	$ A_{\{2,3\}} = 41$
$A_{\{1,2,3\}} = \{120, 240, 360, \dots\}$	$ A_{\{1,2,3\}} = 8$

$$\begin{aligned} |\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3| &= 1000 - (200 + 166 + 125) \\ &\quad + (33 + 25 + 41) - 8 \\ &= 600. \end{aligned}$$

General Case

$$A_1, A_2, \dots, A_N \subseteq A.$$

$$\text{For } S \subseteq [N], \; A_S = \cap_{i \in S} A_i.$$

$$\text{E.g. } A_{\{4,7,18\}} = A_4 \cap A_7 \cap A_{18}.$$

$$A_\emptyset = A.$$

Inclusion-Exclusion Formula:

$$\left| \bigcap_{i=1}^N \bar{A}_i \right| = \sum_{S \subseteq [N]} (-1)^{|S|} |A_S|.$$

Derangements

We must express the set of derangements D_n of $[n]$ as the intersection of the complements of sets.

We let $A_i = \{\text{permutations } \pi : \pi(i) = i\}$ and then

$$D_n = \left| \bigcap_{i=1}^n \bar{A}_i \right|.$$

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We must now compute $|A_S|$ for $S \subseteq [n]$.

$|A_1| = (n-1)!$: after fixing $\pi(1) = 1$ there are $(n-1)!$ ways of permuting $2, 3, \dots, n$.

$|A_{\{1,2\}}| = (n-2)!$: after fixing $\pi(1) = 1, \pi(2) = 2$ there are $(n-2)!$ ways of permuting $3, 4, \dots, n$.

In general

$$|A_S| = (n - |S|)!$$

$$\begin{aligned} |D_n| &= \sum_{S \subseteq [n]} (-1)^{|S|} (n - |S|)! \\ &= \sum_{k=0}^n (-1)^k \binom{n}{k} (n-k)! \\ &= \sum_{k=0}^n (-1)^k \frac{n!}{k!} \\ &= n! \sum_{k=0}^n (-1)^k \frac{1}{k!}. \end{aligned}$$

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Proof of inclusion-exclusion formula

$$\theta_{x,i} = \begin{cases} 1 & x \in A_i \\ 0 & x \notin A_i \end{cases}$$

Then

$$(1 - \theta_{x,1})(1 - \theta_{x,2}) \cdots (1 - \theta_{x,N}) = \begin{cases} 1 & x \in \bigcap_{i=1}^N \bar{A}_i \\ 0 & \text{otherwise} \end{cases}$$

So

$$\begin{aligned} \left| \bigcap_{i=1}^N \bar{A}_i \right| &= \sum_{x \in A} (1 - \theta_{x,1})(1 - \theta_{x,2}) \cdots (1 - \theta_{x,N}) \\ &= \sum_{x \in A} \sum_{S \subseteq [N]} (-1)^{|S|} \prod_{i \in S} \theta_{x,i} \\ &= \sum_{S \subseteq [N]} (-1)^{|S|} \sum_{x \in A} \prod_{i \in S} \theta_{x,i} \\ &= \sum_{S \subseteq [N]} (-1)^{|S|} |A_S|. \end{aligned}$$

Euler's Function $\phi(n)$.

Let $\phi(n)$ be the number of positive integers $x \leq n$ which are mutually prime to n i.e. have no common factors with n , other than 1.

$$\phi(12) = 4.$$

Let $n = p_1^{\alpha_1} p_2^{\alpha_2} p_1^{\alpha_2} \cdots p_k^{\alpha_k}$ be the prime factorisation of n .

$$A_i = \{x \in [n] : p_i \text{ divides } x\}, \quad 1 \leq i \leq k.$$

$$\phi(n) = \left| \bigcap_{i=1}^k \bar{A}_i \right|$$

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Surjections

Fix n, m . Let

$$A = \{f : [n] \rightarrow [m]\}$$

Thus $|A| = m^n$. Let

$$F(n, m) = \{f \in A : f \text{ is onto } [m]\}.$$

How big is $F(n, m)$? – note that

$$S(n, k) = |F(n, k)|/k!.$$

Let

$$A_i = \{f \in F : f(x) \neq i, \forall x \in [n]\}.$$

Then

$$F(n, m) = \bigcap_{i=1}^m \bar{A}_i.$$

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For $S \subseteq [m]$

$$\begin{aligned} A_S &= \{f \in A : f(x) \notin S, \forall x \in [n]\}. \\ &= \{f : [n] \rightarrow [m] \setminus S\}. \end{aligned}$$

So

$$|F_S| = (m - |S|)^n.$$

Hence

$$\begin{aligned} F(n, m) &= \sum_{S \subseteq [m]} (-1)^{|S|} (m - |S|)^n \\ &= \sum_{k=0}^m (-1)^k \binom{m}{k} (m - k)^n. \end{aligned}$$

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