

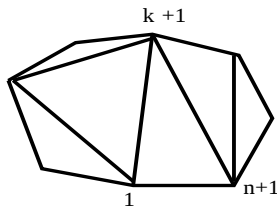
Triangulation of n -gon

Let

$$\begin{aligned} a_n &= \text{no. of triangulations of } P_{n+1} \\ &= \sum_{k=0}^n a_k a_{n-k} \quad n \geq 2 \end{aligned} \quad (1)$$

$$a_0 = 0, \quad a_1 = a_2 = 1.$$

Solution – use generating functions.



Explanation of (1):

$a_k a_{n-k}$ counts the number of triangulations in which edge $1, n+1$ is contained in triangle $1, k+1, n+1$.

There are a_k ways of triangulating $1, 2, \dots, k+1, 1$ and for each such there are a_{n-k} ways of triangulating $k+1, k+2, \dots, n+1, k+1$.

If $a_0, a_1, \dots, a_n$ is a sequence of real numbers then its **(ordinary) generating function** $a(x)$ is given by

$$a(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n + \dots$$

and we write

$$a_n = [x^n] a(x).$$

$$a_n = 1$$

$$a(x) = \frac{1}{1-x} = 1 + x + x^2 + \dots + x^n + \dots$$

$$a_n = n + 1.$$

$$a(x) = \frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \dots + (n+1)x^n + \dots$$

$$a_n = n.$$

$$a(x) = \frac{x}{(1-x)^2} = x + 2x^2 + 3x^3 + \dots + nx^n + \dots$$

$$a_n = \binom{\alpha}{n}$$

$$a(x) = (1+x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n.$$

$$a_n = \binom{c+n-1}{n}$$

$$a(x) = \frac{1}{(1-x)^c} = \sum_{n=0}^{\infty} \binom{-c}{n} (-x)^n = \sum_{n=0}^{\infty} \binom{c+n-1}{n} x^n.$$

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Simple properties of generating functions

- $a_n = [x^n]a(x)$ and $b_n = [x^n]b(x)$.
 $a(x) + b(x) = \sum_{n=0}^{\infty} (a_n + b_n)x^n$ and so
 $a_n + b_n = [x^n]a(x) + b(x)$.

- $a_n = [x^n]a(x)$. Then

$$xa(x) = \sum_{n=0}^{\infty} a_n x^{n+1} = \sum_{n=1}^{\infty} a_{n-1} x^n.$$

So $a_{n-1} = [x^n]xa(x)$.

e.g. $n+1 = [x^n](1-x)^{-2}$ and

$$n = [x^n]x(1-x)^{-2}.$$

- $a_n = [x^n]a(x)$. Then

$$a'(x) = \sum_{n=1}^{\infty} na_n x^{n-1} = \sum_{n=0}^{\infty} (n+1)a_{n+1} x^n$$

and so $(n+1)a_{n+1} = [x^n]a'(x)$.

e.g. $[n+1] = [x^n](1-x)^{-2}$ and so

$$(n+1)(n+2) = [x^n]2(1-x)^{-3}.$$

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So (2) gives

$$a(x) - 1 - 9x - 6x(a(x) - 1) + 9x^2 a(x) = 0$$

or

$$a(x)(1 - 6x + 9x^2) - (1 + 3x) = 0.$$

So

$$\begin{aligned} a(x) &= \frac{1+3x}{1-6x+9x^2} \\ &= \frac{1+3x}{(1-3x)^2} \\ &= \sum_{n=0}^{\infty} (n+1)3^n x^n + 3x \sum_{n=0}^{\infty} (n+1)3^n x^n \\ &= \sum_{n=0}^{\infty} (n+1)3^n x^n + \sum_{n=0}^{\infty} n3^n x^n \\ &= \sum_{n=0}^{\infty} (2n+1)3^n x^n. \end{aligned}$$

So

$$a_n = (2n+1)3^n.$$

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Solution of linear recurrences

$$a_n - 6a_{n-1} + 9a_{n-2} = 0 \quad n \geq 2.$$

$$a_0 = 1, a_1 = 9.$$

$$\sum_{n=2}^{\infty} (a_n - 6a_{n-1} + 9a_{n-2})x^n = 0. \quad (2)$$

$$\sum_{n=2}^{\infty} a_n x^n = a(x) - a_0 - a_1 x$$

$$= a(x) - 1 - 9x.$$

$$\sum_{n=2}^{\infty} 6a_{n-1} x^n = 6x \sum_{n=2}^{\infty} a_{n-1} x^{n-1}$$

$$= 6x(a(x) - a_0)$$

$$= 6x(a(x) - 1).$$

$$\sum_{n=2}^{\infty} 9a_{n-2} x^n = 9x^2 \sum_{n=2}^{\infty} a_{n-2} x^{n-2}$$

$$= 9x^2 a(x).$$

$$a_n - 3a_{n-1} = n^2 \quad n \geq 1.$$

$$a_0 = 1.$$

$$\sum_{n=1}^{\infty} (a_n - 3a_{n-1})x^n = \sum_{n=1}^{\infty} n^2 x^n$$

$$\sum_{n=1}^{\infty} n^2 x^n = \sum_{n=2}^{\infty} n(n-1)x^n + \sum_{n=1}^{\infty} nx^n$$

$$= \frac{2x^2}{(1-x)^3} + \frac{x}{(1-x)^2}$$

$$= \frac{x+x^2}{(1-x)^3}$$

$$\sum_{n=1}^{\infty} (a_n - 3a_{n-1})x^n = a(x) - 1 - 3xa(x)$$

$$= a(x)(1-3x) - 1.$$

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$$\begin{aligned}
 a(x) &= \frac{x+x^2}{(1-x)^3(1-3x)} + \frac{1}{1-3x} \\
 &= \frac{A}{1-x} + \frac{B}{(1-x)^2} + \frac{C}{(1-x)^3} + \frac{D+1}{1-3x}
 \end{aligned}$$

where

$$\begin{aligned}
 x+x^2 &\cong A(1-x)^2(1-3x) + B(1-x)(1-3x) \\
 &\quad + C(1-3x) + D(1-x)^3.
 \end{aligned}$$

Then

$$A = -1/2, B = 0, C = -1, D = 3/2.$$

So

$$\begin{aligned}
 a(x) &= \frac{1/6}{1-x} - \frac{1}{(1-x)^3} + \frac{5/2}{1-3x} \\
 &= \frac{1}{6} \sum_{n=0}^{\infty} x^n - \sum_{n=0}^{\infty} \binom{n+2}{2} x^n + \frac{5}{2} \sum_{n=0}^{\infty} 3^n x^n
 \end{aligned}$$

So

$$\begin{aligned}
 a_n &= \frac{1}{6} - \binom{n+2}{2} + \frac{5}{2} 3^n \\
 &= -\frac{5}{6} - \frac{3n}{2} - \frac{n^2}{2} + \frac{5}{2} 3^n.
 \end{aligned}$$

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$$a(x) = b(x) = 1/(1-x). \quad a_n = b_n = 1, n \geq 0.$$

$$c_n = \sum_{k=0}^n 1 \times 1 = n+1.$$

So

$$\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} (n+1)x^n.$$

$$a(x) = e^{\alpha x}, b(x) = \frac{1}{1-x}.$$

$$c_n = \sum_{k=0}^n \frac{\alpha^k}{k!}.$$

$$\frac{e^{\alpha x}}{1-x} = \sum_{n=0}^{\infty} \left(\sum_{k=0}^n \frac{\alpha^k}{k!} \right) x^n.$$

Products of generating functions

$$a(x) = \sum_{n=0}^{\infty} a_n x^n, b(x) = \sum_{n=0}^{\infty} b_n x^n.$$

$$\begin{aligned}
 a(x)b(x) &= (a_0 + a_1x + a_2x^2 + \dots) \times \\
 &\quad (b_0 + b_1x + b_2x^2 + \dots) \\
 &= a_0b_0 + (a_0b_1 + a_1b_0)x + \\
 &\quad (a_0b_2 + a_1b_1 + a_2b_0)x^2 + \dots \\
 &= \sum_{n=0}^{\infty} c_n x^n
 \end{aligned}$$

where

$$c_n = \sum_{k=0}^n a_k b_{n-k}.$$

Solution of polygon triangulation problem.

$$x + \sum_{n=2}^{\infty} a_n x^n = x + \sum_{n=2}^{\infty} \left(\sum_{k=0}^n a_k a_{n-k} \right) x^n.$$

But,

$$x + \sum_{n=2}^{\infty} a_n x^n = a(x)$$

since $a_0 = 0, a_1 = 1$.

$$\begin{aligned}
 \sum_{n=2}^{\infty} \left(\sum_{k=0}^n a_k a_{n-k} \right) x^n &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^n a_k a_{n-k} \right) x^n \\
 &= a(x)^2.
 \end{aligned}$$

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So

$$a(x) = x + a(x)^2$$

and hence

$$a(x) = \frac{1 + \sqrt{1-4x}}{2} \text{ or } \frac{1 - \sqrt{1-4x}}{2}.$$

But $a(0) = 0$ and so

$$\begin{aligned} a(x) &= \frac{1 - \sqrt{1-4x}}{2} \\ &= \frac{1}{2} - \frac{1}{2} \left(1 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} (2n-2)}{n 2^{2n-1}} \binom{2n-2}{n-1} (-4x)^n \right) \\ &= \sum_{n=1}^{\infty} \frac{1}{n} \binom{2n-2}{n-1} x^n. \end{aligned}$$

So

$$a_n = \frac{1}{n} \binom{2n-2}{n-1}.$$

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Let

$$d(x) = \sum_{m=0}^{\infty} \frac{d_m x^m}{m!}.$$

From (3) we have

$$\begin{aligned} \frac{1}{1-x} &= e^x d(x) \\ d(x) &= \frac{e^{-x}}{1-x} \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^n \left(\frac{(-1)^k}{k!} \right) x^n. \end{aligned}$$

So

$$\frac{d_n}{n!} = \sum_{k=0}^n \frac{(-1)^k}{k!}.$$

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Derangements

$$n! = \sum_{k=0}^n \binom{n}{k} d_{n-k}.$$

Explanation: $\binom{n}{k} d_{n-k}$ is the number of permutations with exactly k cycles of length 1. Choose k elements ($\binom{n}{k}$ ways) for which $\pi(i) = i$ and then choose a derangement of the remaining $n-k$ elements.

So

$$\begin{aligned} 1 &= \sum_{k=0}^n \frac{1}{k!} \frac{d_{n-k}}{(n-k)!} \\ \sum_{n=0}^{\infty} x^n &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^n \frac{1}{k!} \frac{d_{n-k}}{(n-k)!} \right) x^n. \quad (3) \end{aligned}$$

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