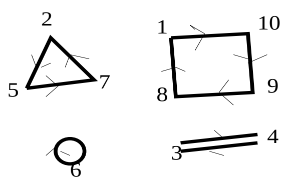
## Cycles of permutations

$\pi : [n] \to [n]$ is a permutation i.e. is 1-1 and onto.

Example:

| $i$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\pi(i)$ | 10 | 5 | 4 | 3 | 7 | 6 | 2 | 1 | 8 | 9 | $\leftarrow$ permutation |

Draw diagram with $n$ points $1, 2, \ldots, n$ and join $i \to \pi(i)$ by a directed edge.




NOT A DERANGEMENT



## Derangements

A *derangement* is a permutation $\pi : [n] \to [n]$ such that $\pi(i) \neq i$ for all $i$.

Equivalently $\pi$ has no cycles of length 1.

$D_n = \{$ derangements of $[n]\}$ and $d_n = |D_n|$.

$d_1 = 0, d_2 = 1, d_3 = 2, d_4 = 9$.

$n = 3:$ 2,3,1 or 3,1,2.

**Claim**: $d_n = (n-1)d_{n-1} + (n-1)d_{n-2}$.



$D_n = D'_n \cup D''_n$.

$$D'_n = \{\pi \in D_n : \ n \text{ lies in a cycle of length 2}\}$$

i.e. $\pi(\pi(n)) = n$.

$$= \bigcup_{k=1}^{n-1} D'_{n,k}$$

$$= \bigcup_{k=1}^{n-1} \{\pi \in D_n : \ \pi(n) = k \text{ and } \pi(k) = n\}.$$

**Claim:** $|D'_{n,k}| = d_{n-2}$

$$f : D'_{n,k} \to D_{n-2}.$$

Action of $f$: remove cycle $(k, n)$ (re-label).
$f$ is a bijection and so

$$|D'_n| = (n-1)d_{n-2}.$$



$$D''_n = D_n \setminus D'_n = \bigcup_{k=1}^{n-1} D''_{n,k}$$

$$= \bigcup_{k=1}^{n-1} \{\pi \in D_n : \ \pi(n) = k\}$$

**Claim:** $|D''_{n,k}| = d_{n-1}$.

$$g : D'_{n,k} \to D_{n-1}.$$

Action of $g$: replace $x \to n \to k$ by $x \to k$.
$g$ is a bijection and so

$$|D''_n| = (n-1)d_{n-1}.$$

$$\begin{aligned}
d_n &= (n-1)d_{n-1} + (n-1)d_{n-2} \\
\frac{d_n}{n!} &= \left(1 - \frac{1}{n}\right)\frac{d_{n-1}}{(n-1)!} + \frac{1}{n}\frac{d_{n-2}}{(n-2)!}.
\end{aligned}$$

$$\begin{aligned}
e_n = d_n/n! &= \left(1 - \frac{1}{n}\right) e_{n-1} + \frac{1}{n} e_{n-2} \\
e_n - e_{n-1} &= -\frac{1}{n}(e_{n-1} - e_{n-2}) \\
&= -\frac{1}{n} \times -\frac{1}{n-1}(e_{n-2} - e_{n-3}) \\
&\vdots \\
&= \frac{(-1)^{n-1}}{n!}(e_1 - e_0) \\
e_n - e_{n-1} &= \frac{(-1)^n}{n!} \\
e_{n-1} - e_{n-2} &= \frac{(-1)^{n-1}}{(n-1)!} \\
&\vdots \\
e_1 - e_0 &= -1
\end{aligned}$$

So

$$\begin{aligned}
e_n - e_0 &= (-1)^n \left(\frac{1}{n!} - \frac{1}{(n-1)!} + \cdots + (-1)^{n-1}\right). \\
e_n &= 1 - \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{(-1)^n}{n!} \\
d_n &= n! \left(1 - \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{(-1)^n}{n!}\right) \\
&\approx n! e^{-1} \qquad \text{as } n \to \infty
\end{aligned}$$