Recurrences for Partitions

Bell Numbers: B_n is the number of distinct partitions of $[n]$.
$B_3 = 5$.

$$\{1,2,3\} \quad \{1\},\{2,3\} \quad \{2\},\{1,3\} \quad \{3\},\{1,2\}$$

$$\{1\},\{2\},\{3\}$$

$$B_n = \sum_{k=1}^{n} \binom{n-1}{k-1} B_{n-k}. \qquad (1)$$

$\binom{n-1}{k-1}B_{n-k}$ is the number of partitions of $[n]$ in which n belongs to a set Y of size k – choose $Y \setminus \{k\}$, ($\binom{n-1}{k-1}$ ways), partition $[n] \setminus Y$ (B_{n-k} ways.)

Unfortunately, (1) is hard to solve.

Stirling Numbers of the Second Kind

$\mathcal{P}(n,k) = \{$partitions of $[n]$ into k non-empty parts$\}$.
$S(n,k) = |\mathcal{P}(n,k)|$.
$S(4,2) = 7$:

$$\{1,2\},\{3,4\} \quad \{1,3\},\{2,4\} \quad \{1,4\},\{2,3\}$$

$$\{1\},\{2,3,4\} \quad \{2\},\{1,3,4\} \quad \{3\},\{1,2,4\}$$

$$\{4\},\{1,2,3\}$$

$$S(n,k) = kS(n-1,k) + S(n-1,k-1). \qquad (2)$$

- $S(n-1,k-1)$: no. of partitions $\mathcal{P}_1(n,k)$ in which n occurs on its own as $\{n\}$.
- $kS(n-1,k)$: no. of partitions $\mathcal{P}_2(n,k)$ in which n occurs in a set Y of size ≥ 2.

$$f : \mathcal{P}_2(n,k) \to \mathcal{P}(n-1,k).$$

f removes n from the set Y containing it.

f is onto and each partition P in $\mathcal{P}(n-1,k)$ is the image of k members of $\mathcal{P}_2(n,k)$ – going back from P there are k places to put n.

Unfortunately, (2) is also hard to solve.

Aside: let $(x)_k = x(x-1)\cdots(x-k+1)$. Then

$$x^n = \sum_{k=0}^{n} S(n,k)(x)_k. \qquad (3)$$

Ex.
$x^4 = x(x-1)(x-2)(x-3) + 6x(x-1)(x-2) + 7x(x-1) + x.$

Need only show (3) for $x = 0,1,2,\ldots,n$. For then the degree n polynomial $x^n - \sum_{k=0}^{n} S(n,k)(x)_k$ has $n+1$ roots and so is identically zero.

$x = m \le n$: $m^n = |[m]^{[n]}|$ where $|[m]^{[n]}|$ is the set of functions from $[n]$ to $[m]$.

For $S \subseteq [m]$, $|S| = k$ there are $k!S(n,k)$ functions with $f([n]) = S$.
$S = \{s_1, s_2, \ldots, s_k\}$. Choose partition $P = X_1, X_2, \ldots, X_k$ of $[n]$ – $S(n,k)$ ways. Choose permutation π of $[k]$ – $k!$ ways. Put $f(X_i) = s_{\pi(i)}$, $1 \le i \le k$.

So

$$\begin{aligned} m^n &= \sum_{k=1}^{m} \sum_{S \subseteq [m], |S|=k} k!S(n,k) \\ &= \sum_{k=1}^{m} k!\binom{m}{k} S(n,k) \\ &= \sum_{k=1}^{m} (m)_k S(n,k) \\ &= \sum_{k=0}^{m} (m)_k S(n,k) \qquad S(n,0) = 0. \end{aligned}$$