

Recurrences for Partitions

Bell Numbers: B_n is the number of distinct partitions of $[n]$.

$$B_3 = 5.$$

$$\{1, 2, 3\} \quad \{1\}, \{2, 3\} \quad \{2\}, \{1, 3\} \quad \{3\}, \{1, 2\} \\ \{1\}, \{2\}, \{3\}$$

$$B_n = \sum_{k=1}^n \binom{n-1}{k-1} B_{n-k}. \quad (1)$$

$\binom{n-1}{k-1} B_{n-k}$ is the number of partitions of $[n]$ in which n belongs to a set Y of size k – choose $Y \setminus \{k\}$, ($\binom{n-1}{k-1}$ ways), partition $[n] \setminus Y$ (B_{n-k} ways.)

Unfortunately, (1) is hard to solve.

Stirling Numbers of the Second Kind

$\mathcal{P}(n, k) = \{\text{partitions of } [n] \text{ into } k \text{ non-empty parts}\}.$

$$S(n, k) = |\mathcal{P}(n, k)|.$$

$$S(4, 2) = 7 :$$

$$\{1, 2\}, \{3, 4\} \quad \{1, 3\}, \{2, 4\} \quad \{1, 4\}, \{2, 3\} \\ \{1\}, \{2, 3, 4\} \quad \{2\}, \{1, 3, 4\} \quad \{3\}, \{1, 2, 4\} \\ \{4\}, \{1, 2, 3\}$$

$$S(n, k) = kS(n-1, k) + S(n-1, k-1). \quad (2)$$

- $S(n-1, k-1)$: no. of partitions $\mathcal{P}_1(n, k)$ in which n occurs on its own as $\{n\}$.
- $kS(n-1, k)$: no. of partitions $\mathcal{P}_2(n, k)$ in which n occurs in a set Y of size ≥ 2 .

$$f : \mathcal{P}_2(n, k) \rightarrow \mathcal{P}(n-1, k).$$

f removes n from the set Y containing it.

f is onto and each partition P in $\mathcal{P}(n-1, k)$ is the image of k members of $\mathcal{P}_2(n, k)$ – going back from P there are k places to put n .

Unfortunately, (2) is also hard to solve.

Aside: let $(x)_k = x(x-1) \cdots (x-k+1)$. Then

$$x^n = \sum_{k=0}^n S(n, k)(x)_k. \quad (3)$$

Ex.

$$x^4 = x(x-1)(x-2)(x-3) + 6x(x-1)(x-2) + 7x(x-1) + x.$$

Need only show (3) for $x = 0, 1, 2, \dots, n$. For then the degree n polynomial $x^n - \sum_{k=0}^n S(n, k)(x)_k$ has $n+1$ roots and so is identically zero.

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$x = m \leq n$: $m^n = |[m]^{[n]}|$ where $|[m]^{[n]}|$ is the set of functions from $[n]$ to $[m]$.

For $S \subseteq [m]$, $|S| = k$ there are $k!S(n, k)$ functions with $f([n]) = S$.

$S = \{s_1, s_2, \dots, s_k\}$. Choose partition $P = X_1, X_2, \dots, X_k$ of $[n] - S(n, k)$ ways. Choose permutation π of $[k] - k!$ ways. Put $f(X_i) = s_{\pi(i)}$, $1 \leq i \leq k$.

So

$$\begin{aligned}
 m^n &= \sum_{k=1}^m \sum_{S \subseteq [m], |S|=k} k!S(n, k) \\
 &= \sum_{k=1}^m k! \binom{m}{k} S(n, k) \\
 &= \sum_{k=1}^m (m)_k S(n, k) \\
 &= \sum_{k=0}^m (m)_k S(n, k) \quad S(n, 0) = 0.
 \end{aligned}$$

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