Two applications in Computer Science

Sorting. Assume we are given n boxes, $B_1, B_2, \ldots, B_n$. Box B_i contains a *distinct* number x_i which we are not allowed to see.

In each step, we are allowed to *compare* the contents of two boxes B_i, B_j of our choice i.e. determine whether $x_i < x_j$ or $x_i > x_j$.

How many comparisons do we have to do in order to be able to *sort* the boxes into increasing order of their contents?

More precisely let

$$C(A, I) \;=\; \text{number of comparisons of algorithm } A \text{ on instance } I$$

and

$$T(n) = \min_{A} \max_{I:|I|=n} C(A, I).$$

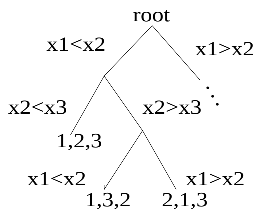
Theorem 1

$$T(n) = n \log_2 n + O(n)$$

i.e. there exist c_1, c_2 such that

$$n \log_2 n - c_1 n \le T(n) \le n \log_2 n + c_2 n.$$

Proof We represent each algorithm A by a *binary decision tree* $T(A)$:



In the above algorithm we first compare x_1, x_2. If $x_1 < x_2$ we compare x_2, x_3 and so on.

A leaf v of the tree (a vertex with no descendants) is labelled by the order implied by the path from the root to v. Thus in the tree above, the leftmost leaf is labelled by 1,2,3 since the labels on the edges to the root are $x_1 < x_2$ and $x_2 < x_3$.

We can assume that for every path of the tree there is a (unique) ordering of the boxes which "satisfies" the labels of the path, otherwise we can remove some edges of the tree.

The *height* $h(T)$ of a tree T is the maximum number of edges in a path from the root to a leaf. The *depth* of a vertex v is the number of edges in the path from the root to v.

$$\max_I C(A, I) = h(T(A)). \tag{1}$$

We will prove the following

Lemma 1 *A binary tree of height h has at most 2^h leaves.*

Now for any A, $T(A)$ has $n!$ leaves, one for each possible ordering of the boxes. Lemma 1 implies then that for all A,

$$\begin{aligned} h(T(A)) &\ge \log_2(n!) \ge \log_2((n/e)^n) \\ &= n \log_2 n - O(n). \end{aligned}$$

Thus

$$T(n) \ge n \log_2 n - O(n).$$

Proof of Lemma 1 By induction on the height h. It is simple for $h=1$ and so assume that any tree of height $h' < h$ has at most $2^{h'}$ leaves and let T be a binary tree of height h. Suppose it has k leaves.

Delete all the leaves of T of depth h and the edges incident with them. We obtain a tree T' of height $h-1$ which has k_1 leaves which are also leaves of T and k_2 leaves which are immmediate ancestors of leaves of T of height h. The result follows from

$$k_1 + k_2 \le 2^{h-1} \text{ and } k \le k_1 + 2k_2.$$

$\square$

Merge Sort

$MS(n)$
If $n \ge 2$;
Partition boxes into 2 sets of size $\lfloor n/2 \rfloor, \lceil n/2 \rceil$;
Apply $MS(\lfloor n/2 \rfloor)$ to first set to get sorted list L_1;
Apply $MS(\lceil n/2 \rceil)$ to second set to get sorted list L_2;
MERGE the 2 sorted lists:
Create sorted list L as follows:
Repeatedly delete the minimum x of $front(L_1), front(L_2)$ until L_1, L_2 are both empty;
Place x at the back of list L

At the end of this procedure, L contains the boxes in sorted order.

Let $C(n)$ be the maximum number of comparisons that $MS(n)$ uses to sort n elements. It is not easy to see what this is directly, instead we set up a recurrence.

$C(1) = 0$ and

$$C(n) \le C(\lfloor n/2 \rfloor) + C(\lceil n/2 \rceil) + n - 1. \quad (2)$$

Lemma 2

$$C(n) \le n \log_2 n + n$$

Proof By induction on n. This clearly true for $n=1$. Assume $n \ge 2$.

Case 1: $n = 2m$ is even.

Applying (2) we see that

$$\begin{aligned} C(n) &\le 2(m \log_2 m + m) + n - 1 \\ &= n \log_2 n + n - 1. \end{aligned}$$

Case 2: $n = 2m+1$ is odd. Applying (2) we see that

$$\begin{aligned} C(n) &\le m \log_2 m + m + (m+1)\log_2(m+1) \\ &\qquad + m + 1 + n - 1 \\ &= n \log_2 n + n - \alpha_n. \end{aligned}$$

where

$$\begin{aligned} \alpha_n &= 1 - \tfrac{n-1}{2} \log_2(1 - \tfrac{1}{n}) - \tfrac{n+1}{2} \log_2(1 + \tfrac{1}{n}) \\ &= 1 - (\log_e 2)^{-1} \sum_{k=0}^{\infty} \frac{n^{-(2k+1)}}{(2k+1)(2k+2)} \\ &\ge 0. \end{aligned}$$

$\square$

Fast multiplication. Assume, that we have to multiply two decimal numbers, both are of length n. If we multiply them digit by digit, then we have to make n^2 (digit-by-digit) multiplications. How can we reduce this number for large n?

Assume that $n = 2^k$ for some integer k and that the two numbers are $a = 10^{n/2}a_1 + a_2$, and $b = 10^{n/2}b_1 + b_2$ where a_1, a_2, b_1, b_2 are all $n/2$ digit numbers. Then

$$\begin{aligned} ab &= 10^n a_1 b_1 + 10^{n/2}(a_1 b_2 + a_2 b_1) + a_2 b_2 \\ &= 10^n a_1 b_1 + \\ &\quad 10^{n/2}((a_2 + a_1)(b_1 + b_2) - a_1 b_1 - a_2 b_2) \\ &\quad + a_2 b_2 \end{aligned}$$

What has been gained?

Let $M(n)$ denote the total number of 2 digit multiplications and additions needed if we carry out the computation as indicated. Then

$$M(n) \leq 3M(n/2) + 4n.$$

$3M(n/2)$ comes from the 3 multiplications, $a_1 b_1, a_2 b_2$ and $(a_2 + a_1)(b_1 + b_2)$.

The $4n$ comes from the remaining additions.

Multiplying by a power of 10 is ignored, and only involves $O(n)$ work.

Let $a_k = M(2^k)$ so that $a_0 = M(1) = 1$ and

$$a_k \leq 3a_{k-1} + 2^{k+2}.$$

Dividing by 3^k we get

$$\begin{aligned} \frac{a_k}{3^k} &\leq \frac{a_{k-1}}{3^{k-1}} + 4 \times \left(\frac{2}{3}\right)^k \\ \frac{a_{k-1}}{3^{k-1}} &\leq \frac{a_{k-2}}{3^{k-2}} 4 \times \left(\frac{2}{3}\right)^{k-1} \\ &\vdots \\ \frac{a_1}{3} &\leq a_0 + 4 \end{aligned}$$

So

$$\begin{aligned} \frac{a_k}{3^k} - 1 &\leq 4\left(1 + \frac{2}{3} + \cdots + \left(\frac{2}{3}\right)^k\right) \\ &< \frac{4}{1 - \frac{2}{3}} = 12. \end{aligned}$$

So

$$\begin{aligned} a_k &\leq 13 \times 3^k = 13 \times 3^{\log_2 n} \\ M(n) &\leq 13 \times n^{\log_2 3}. \end{aligned}$$

If n is not a power of 2 then we can pad it out with $< n$ zeros to make it one. Thus

$$M(n) = O(n^{\log_2 3}).$$