Inequalities

Markov Inequality: let $X : \Omega \to \{0, 1, 2, \ldots ,\}$ be a random variable. For any $t \geq 1$

$$\mathbf{P}(X \geq t) \leq \frac{\mathbf{E}(X)}{t}.$$

Proof

$$\begin{aligned} \mathbf{E}(X) &= \sum_{k=0}^{\infty} k\mathbf{P}(X = k) \\ &\geq \sum_{k=t}^{\infty} k\mathbf{P}(X = k) \\ &\geq \sum_{k=t}^{\infty} t\mathbf{P}(X = k) \\ &= t\mathbf{P}(X \geq t). \end{aligned}$$

In particular, if $t = 1$ then

$$\mathbf{P}(X \neq 0) \leq \mathbf{E}(X).$$

m distinguishable balls, n boxes

Z = number of empty boxes.

$$m \geq (1 + \epsilon)n \log_e n.$$

$$\begin{aligned} \mathbf{E}(Z) &= n\left(1 - \frac{1}{n}\right)^m \\ &\leq ne^{-m/n} \\ &\leq ne^{-(1+\epsilon)\log_e n} \\ &= n^{-\epsilon}. \end{aligned}$$

So

$$\mathbf{P}(\exists \text{ an empty box}) \leq n^{-\epsilon}.$$

Variance:

$Z : \Omega \to \mathbf{R}$ and $\mathbf{E}(Z) = \mu$.

$$\begin{aligned} \mathbf{Var}(Z) &= \mathbf{E}((Z - \mu)^2) \\ &= \mathbf{E}(Z^2 - 2\mu Z + \mu^2) \\ &= \mathbf{E}(Z^2) - \mathbf{E}(2\mu Z) + \mathbf{E}(\mu^2) \\ &= \mathbf{E}(Z^2) - 2\mu\mathbf{E}(Z) + \mu^2 \\ &= \mathbf{E}(Z^2) - \mu^2. \end{aligned}$$

Ex. Two Dice. $\mathrm{Z}(x_1, x_2) = x_1 + x_2$.

$\mathbf{Var}(\mathrm{Z}) = \frac{2^2\times 1}{36} + \frac{3^2\times 2}{36} + \frac{4^2\times 3}{36} + \frac{5^2\times 4}{36} + \frac{6^2\times 5}{36} + \frac{7^2\times 6}{36} + \frac{8^2\times 5}{36} + \frac{9^2\times 4}{36} + \frac{10^2\times 3}{36} + \frac{11^2\times 2}{36} + \frac{12^2\times 1}{36} - 7^2 = \frac{35}{6}$

Binomial: $Z = B_{n,p}$, $\mu = np$.

$$\begin{aligned} \mathbf{Var}(B_{n,p}) &= \sum_{k=1}^{n} k^2 \binom{n}{k} p^k (1-p)^{n-k} - \mu^2 \\ &= \sum_{k=2}^{n} k(k-1) \binom{n}{k} p^k (1-p)^{n-k} + \mu - \mu^2 \\ &= n(n-1)p^2 \sum_{k=2}^{n} \binom{n-2}{k-2} p^{k-2} (1-p)^{n-k} + \mu - \mu^2 \\ &= n(n-1)p^2 (p + (1-p))^{n-2} + \mu - \mu^2 \\ &= n(n-1)p^2 + \mu - \mu^2 \\ &= np(1-p). \end{aligned}$$

Chebycheff Inequality

Now let $\sigma = \sqrt{\mathbf{Var}(Z)}$.

$$\begin{aligned} \mathbf{P}(|Z-\mu| \geq t\sigma) &= \mathbf{P}((Z-\mu)^2 \geq t^2\sigma^2) \\ &\leq \frac{\mathbf{E}((Z-\mu)^2)}{t^2\sigma^2} \qquad (1) \\ &= \frac{1}{t^2}. \end{aligned}$$

(1) comes from the Markov inequality applied to the random variable $(Z-\mu)^2$.

Back to Binomial: $\sigma = \sqrt{np(1-p)}$.

$$\mathbf{P}(|B_{n,p} - np| \geq t\sqrt{np(1-p)}) \leq \frac{1}{t^2}$$

which implies

$$\mathbf{P}(|B_{n,p} - np| \geq \epsilon np) \leq \frac{1}{\epsilon^2 np}.$$

[Law of large numbers.]

If X and Y are **independent** random variables then

$$\mathbf{E}(XY) = \mathbf{E}(X)\mathbf{E}(Y).$$

$$\begin{aligned} \mathbf{E}(XY) &= \sum_\alpha \sum_\beta \alpha\beta \mathbf{P}(X=\alpha, Y=\beta) \\ &= \sum_\alpha \sum_\beta \alpha\beta \mathbf{P}(X=\alpha)\mathbf{P}(Y=\beta) \\ &= \left[\sum_\alpha \alpha \mathbf{P}(X=\alpha)\right]\left[\sum_\beta \beta \mathbf{P}(Y=\beta)\right] \\ &= \mathbf{E}(X)\mathbf{E}(Y). \end{aligned}$$

This is not true if X and Y are not independent. E.g. Two Dice: $X = x_1 + x_2$ and $Y = x_1$.
$\mathbf{E}(X) = 7$, $\mathbf{E}(Y) = 7/2$ and $\mathbf{E}(XY) = \mathbf{E}(x_1^2) + \mathbf{E}(x_1 x_2) = 91/6 + (7/2)^2$.