Introduction

k indistinguishable balls, each given one of n distinct colours.

$$f(n,k) = \# \text{ possible colourings.}$$

Ex. $n = k = 3$

3R	2R+1B	2R+1W
3B	2B+1R	2B+1W
3W	2W+1R	2W+1B

1R+1B+1W

$$f(3,3) = 10.$$

Alternatively, if x_i denotes the number of balls coloured i then

$$x_1 + x_2 + \cdots x_n = k$$

and $f(n,k)$ is the number of non-negative integer solutions to the above equation.

Special Cases:

- $f(1,k) = 1$
- $f(n,1) = n$
- $f(2,k) = k+1$

General approach needed to find $f(n,k)$

Approach 1: *Recurrence*

$$f(n,k) = f(n-1,k) + f(n,k-1). \qquad (1)$$

- nth colour not used: $f(n-1,k)$ ways.
- nth colour used: $f(n,k-1)$ ways.

Given $f(1,k) = 1$ and $f(n,1) = n$ for all n,k we can use (1) to compute $f(n,k)$ for any k,n.

More examples of recurrence relations:

Fibonacci sequence: 1,1,2,3,5,8,13,21,34,55,...

$$a_0 = 1, a_1 = 1 \qquad \textit{boundary condition}$$
$$a_n = a_{n-1} + a_{n-2}.$$

a_n is number of rabbits at the end of n periods. Each rabbit born in period $n-2$ starts producing rabbits, one per period, when it is 2 periods old.

Simpler example: Suppose $a_1 = 1$ and

$$\begin{aligned} a_{n+1} &= na_n \\ &= n(n-1)a_{n-1} \\ &\vdots \\ &= n(n-1)(n-2)\ldots 2a_1 \\ &= n! \end{aligned}$$

Approach 2: *Generating Functions*

Consider $(1-x)^{-n} = (1+x+x^2+\cdots)(1+x+x^2+\cdots)\ldots(1+x+x^2+\cdots)$.

What is the coefficient of x^k?

Each term is obtained by taking x^{t_1} from the first bracket, taking x^{t_2} from the second bracket, ..., taking x^{t_n} from the nth bracket so that $t_1+t_2+\cdots+t_n = k$.

Thus this coefficient is $f(n,k)$ and we write

$$\begin{aligned} f(n,k) &= [x^k](1-x)^{-n} \\ &= [x^k](1+nx+\frac{n(n+1)}{2}x^2\cdots) \end{aligned}$$

Approach 3: Injective Mapping:

Put k X's and $n-1$ O's in a line:

$$XXOXOXOOX$$

Corresponds to $x_1 = 2, x_2 = 1, x_3 = 1, x_4 = 0, x_5 = 1$. In general there is a 1-1 corespondence between

{colourings of balls}
and
{sequences of k X's and $n-1$ O's}.

Number of sequences of k X's and $n-1$ O's is number of ways of choosing k positions (for the X's) from $n-1+k$ positions or

$$\binom{n-1+k}{k}$$