DISCRETE MATHEMATICS 21228 — MIDTERM

JC

Due in class Wednesday October 15. You may not collaborate but may consult any books or papers you wish, and may also consult with me.

You may work on this exam during any continuous 48 hour period of your choosing between now and the due date.

Please write the name of your recitation instructor and the time and place of your recitation at the top of your midterm.

(1) (10 points) Prove that any infinite poset must either contain an infinite chain or an infinite antichain.

(2) (10 points) Prove that the number of sequences of integers $a_1 < a_2 < \dots a_{n-1}$ with $1 \leq a_i \leq 2i$ is the Catalan number C_n.

(3) (10 points) (A polyandrous version of the Hall marriage theorem). We are given finite sets of boys and girls, and each set X of girls collectively likes at least $2|X|$ many boys. Prove that we can marry off each girl with two boys (with no two girls married to any one boy).

Hint: try replacing this situation with one where you can apply the ordinary Hall marriage theorem.

(4) (20 points) (Ramsey theory for triples) For any set X let $[X]^3$ be the set $\{A \subseteq X : |A| = 3\}$. Prove that for any natural numbers a and b there exists a natural number $N \geq 3$ with the following property: for any set of X of size N and any function f from $[X]^3$ to $\{red, blue\}$, there is either a subset of X of size a from which every triple gets coloured red or a subset of size b which every triple gets coloured blue.

Hint: try induction as for the finite Ramsey theorem, and ask yourself what you might use in place of the pigeonhole principle.

If we define $R_3(a, b)$ be the least such N then find an upper bound for $R_3(a, b)$ as a function of a and b.

(5) (15 points) Let $n > 0$. Find a formula for the number of walks from $(0, 0)$ to $(n, 2n)$ which never go below the line $y = 2x$.

(6) Use the probabilistic method to find a lower bound for $R_3(n, n)$.