DISCRETE MATHEMATICS 21228 — HOMEWORK 9

JC

Due in class Wednesday November 19. You may collaborate but *must* write up your solutions by yourself.

Late homework will not be accepted. Homework must either be typed or written legibly in blue or black ink on alternate lines, illegible homework will be returned ungraded (so you can rewrite it legibly).

Please write the name of your recitation instructor and the time and place of your recitation at the top of your homework.

IMPORTANT: We are now following the convention that all graphs are finite unless explicitly stated to be infinite.

(1) Let a and b be distinct vertices in a connected graph G. An *Eulerian trail* from a to b is a trail (a walk without repeated edges) from a to b which visits each edge exactly once. Prove that such a trail exists if and only if a and b are the only vertices of odd degree in G.

Consider a graph H obtained from G by adding a new vertex u and edges au and ub. Now G has an Eulerian trail from a to $b \iff H$ has an Eulerian cycle $\iff$ all vertices of H have even degree in $H \iff$ only a and b have odd degree in G.

(2) For a graph H let $p_H(n)$ be the number of vertex colourings of H using colours $1, ...n$. Find a formula for $p_H(n)$ for the following graphs H:

(a) The complete graph K_m (the graph with m vertices where every pair is joined by an edge).

By an easy counting argument there are $n(n-1)\dots(n-m+1)$. Note that this formula works even for $n < m$, giving the correct answer of zero colourings.

(b) The m-cycle C_m.

For notational simplicity let $f_m(n) = p_{C_m}(n)$. Obviously $f_3(n) = n(n-1)(n-2)$ because $C_3 = K_3$. For the purposes of this problem it is convenient to make the convention that a "2-cycle" is two points joined by an edge, so that $f_2(n) = n(n-1)$.

Let $m \geq 4$ and let $v_1, \dots v_m$ be the vertices, where the edges are $v_i v_{i+1}$ (for $i < m$) and $v_m v_1$
Consider the vertices v_1 and v_3. A colouring which gives them different colours can be considered as a colouring of an $m-1$ cycle formed by the vertices $v_1, v_3, \dots v_m$ together with a choice of an appropriate colour for v_2. So there are $(n-2)f_{m-1}(n)$ such colourings using n colours.
Similarly a colouring which gives v_1 and v_3 the same colour can be considered as a colouring of an $m-2$ cycle formed by $v_1, v_4, \dots v_m$, together with the choice of an appropriate colour for v_2. So there are $(n-1)f_{m-2}(n)$ colourings of this sort using n colours.
This gives us the recursion

$$f_m(n) = (n-2)f_{m-1}(n) + (n-1)f_{m-2}(n)$$

The first few values are $f_2(n) = n(n-1)$, $f_3(n) = n(n-1)(n-2)$, $f_4(n) = n(n-1)(n^2-3n+3)$, $f_5(n) = n(n-1)(n-2)(n^2-2n+2)$.
We can obtain a closed form for this expression as follows. Start by noticing that $f_3(n) = (n-1)(n^2-2n)$, $f_4(n) = (n-1)(n^3-3n^2+3n)$, $f_5(n) = (n-1)(n^4-4n^3+6n^2-4n)$. At this point we recognise the binomials and make the conjecture that $f_m(n) = (n-1)((n-1)^{m-1} + (-1)^m)$. This can now be verified using the recursion we gave above.
Cultural note: let G by any graph and let a and b be vertices not joined by an edge. Then if $G_0 = G + ab$ and G_1 is the graph obtained by identifying the vertices a and b, we can see by considerations as above that $p_G(n) = p_{G_0}(n) + p_{G_1}(n)$. With a little thought this can be used to show that $p_G(n)$ is always a polynomial in n with integer coefficients.

(c) The graph obtained from a 6-cycle by joining two opposite vertices.
Start by noticing that if we are colouring C_4 and we prescribe the colours of two adjacent points, then there are $n^2 - 3n + 3$ ways of colouring the other two points. Either use the formula for $p_{C_4}(n)$ and symmetry or do it directly. The graph is obtained by taking two copies of C_4 and gluing them together. Let ab be the common edge. Then there are $n(n-1)$ ways of colouring ab and $n^2 - 3n + 3$ ways of colouring the rest of each copy of C_4. This gives a total of $n(n-1)(n^2-3n+3)^2$ many colourings.

The work we did before gives us an easy check: by the first argument we gave for C_m, we should have the equation $p_{C_6}(n) = p_{H_0}(n) + p_{H_1}(n)$ where H_0 is the graph from this question and H_1 is the graph in which two triangles are joined at a vertex. Easily $p_{H_1}(n) = n(n-1)^2(n-2)^2$. Let us compare values when $n = 11$. We have easily $p_{C_6}(10) = 1000010, p_{H_0}(11) = 910910, p_{H_1}(11) = 89100$. Indeed $1000010 = 910910 + 89100$, giving us some faith in our answer.

(d) The tree with vertices $\{a, b, c, d, e\}$ and edges $\{ab, ac, bd, be\}$. $n(n-1)^4$.

(3) Prove that a graph is planar if and only if it can be drawn on the surface of a sphere.

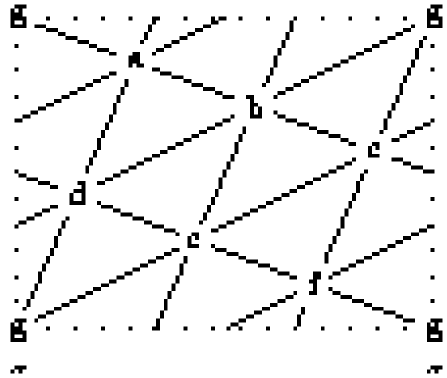
Working in $\mathbb{R}^3$ consider a spherical surface S of radius 1 with its centre at $(0, 0, 1)$. Formally S is given by $x^2+y^2+(z-1)^2 = 1$. Let $N = (0, 0, 2)$ (that is the "north pole" of S) and let P be the plane $z = 0$ (that is the "xy-plane"). Define a function f from $S \setminus \{N\}$ to P as follows: $f(M)$ is the unique point where the line NM meets the plane P.

Now let G be a planar graph and fix a picture of G in the plane P. Then applying f^{-1} we get a picture of G on the surface S. Conversely if we have a picture of a graph G on the surface S, then we may move it so that it does not involve the point N and then apply f to get a picture in the plane P.

(4) (Tricky) Find a graph which can be drawn on the surface of a torus with chromatic number greater than four. Extra credit if you can achieve the optimal value of seven!

Surprisingly we can even draw a K_7 on the surface of a torus. To see this notice that we can make a torus by starting with a square, rolling it up to get a cylinder and then joining the ends of the cylinder: mathematicians typically express this by saying we are "identifying opposite edges" (you may like to think of the old arcade game Asteroids, where objects which went off one side of the screen came back on the opposite side).

Now contemplate the following picture (sorry it is so pixellated!)



(5) Show that a graph G has at least $\binom{\chi(G)}{2}$ edges.

Let $n = \chi(G)$, and fix a n-colouring c of the vertices of G using colours $1, \dots n$. We claim that for every pair $\{i, j\}$ of colours there is an edge ab with $c(a) = i$ and $c(b) = j$. Suppose for a contradiction this is not so; we may as well assume that there is no edge joining a vertex of colour 1 to a vertex of colour n. Define a new function d as follows: $c(v) < n \implies c(v) = d(v)$, $c(v) = n \implies d(v) = 1$. d is an $n - 1$-colouring, contradiction!

So there are at least $\binom{n}{2}$ edges in G.