DISCRETE MATHEMATICS 21228 — HOMEWORK 8 SOLUTIONS

JC

(1) Prove that every tree with more than one vertex has at least two vertices of degree one.

A tree is connected so there are no vertices of degree zero. Suppose for a contradiction that there are v vertices and $v-1$ have degree at least two. Then the sum of the degrees of the vertices is at least $1+2(v-1) = 2v-1$, so the number of edges (which is always one half the sum of the degrees) is at least $v - 1/2$. This is impossible as a tree has exactly $v-1$ edges.

(2) Prove that any connected graph on n vertices has at least $n-1$ edges.

Form a spanning subtree using the algorith from class. The spanning subtree has exactly $n-1$ edges so the original graph has at least $n-1$.

(3) How many edges are there in a forest with n vertices and k connected components?

Each of the k components is a tree, say component i has v_i vertices and $v_i - 1$ edges. o in total there are $\sum_i (v_i - 1) = n - k$ edges.

(4) Let $G = (V, E)$ be a graph, and define $\bar{G} = (V, [V]^2 \setminus E)$. Show that at least one of G and $\bar{G}$ is connected.

Suppose G is not connected, so G has at least two components. Now let v and w be two vertices. If they are in different components of G they are joined by an edge of $\bar{G}$, so are in the same component of $\bar{G}$. If v and w are in the same component of G choose a vertex z belonging to a different component, then vz and zw are both edges of $\bar{G}$ so again v and w are in the same component of $\bar{G}$.

(5) Recall how we proved the upper bound on the number e of edges in a triangle-free graph on n vertices: we showed that $d(x) + d(y) \leq n$ for every edge xy, noticed that in any graph $\sum_{x \in V} d(x) = 2e$ and $\sum_{xy \in E} (d(x) + d(y)) = \sum_{x \in V} d(x)^2$, and then used the inequality $(a_1 + \ldots a_n)^2 \leq n(a_1^2 + \ldots a_n^2)$ with a_i

equal to the degree of the i^{th} vertex to show $4e^2 \leq n^2 e$ where e is the number of edges.

(a) Show that in the inequality $2AB \leq A^2 + B^2$ equality holds exactly when $A = B$.
$2AB = A^2 + B^2$ iff $(A - B)^2 = 0$ iff $A = B$.

(b) Show that in the inequality $(a_1 + \ldots a_n)^2 \leq n(a_1^2 + \ldots a_n^2)$ equality holds exactly when all the a_i are equal.
Recall the proof of the inequality:

$$(\sum_i a_i)^2 = \sum_i a_i^2 + \sum_{i<j} 2a_i a_j \leq \sum_i a_i^2 + \sum_{i<j} (a_i^2 + a_j^2) = n \sum_i a_i^2.$$

Now clearly equality holds exactly when $2a_i a_j = a_i^2 + a_j^2$, that is all the a_i are equal.

(c) Let n be even. Show that in a triangle free graph with n vertices and $n^2/4$ edges, every vertex has degree $n/2$.
By the previous part and the argument for the upper bound sketched in the question, all vertices have the same degree: the sum of the degrees is twice the number of edges so all vertices have degree $n/2$.

(d) (Tricky) Is it true in general that every triangle-free graph on an even number n of vertices with $n^2/4$ edges is bipartite?
After drawing a few pictures we conjecture that for every even n, all triangle-free graphs on n vertices with $n^2/4$ edges are bipartite with $n/2$ vertices on each side and all possible edges between the two sides.
We will prove this by induction on n. The $n = 2$ case is easy :-)
Suppose we have it up to n. Let G be a triangle-free graph on $n + 2$ vertices with $n^2/4 + n + 1$ many edges. Choose v and w with vw an edge of G, and build a new graph H by deleting v and w along with all incident edges. Since v and w each have degree $(n + 2)/2$ we lose a total of $n + 1$ edges, so H is triangle free on n vertices with $n^2/4$ edges.
By induction H is bipartite. A little thought shows that if A is the set of vertices joined to v and B is the set joined to w, then $|A| = |B| = n/2$ and H contains every edge joining a vertex of A to a vertex of B.
So now if we let $A^* = A \cup \{w\}$, $B^* = B \cup \{v\}$, G contains every edge joining A^* to B^* and we are done.