DISCRETE MATHEMATICS 21228 — HOMEWORK 7 SOLUTIONS

JC

Due in class Wednesday October 29. You may collaborate but *must* write up your solutions by yourself.

Late homework will not be accepted. Homework must either be typed or written legibly in blue or black ink on alternate lines, illegible homework will be returned ungraded (so you can rewrite it legibly).

Please write the name of your recitation instructor and the time and place of your recitation at the top of your homework.

(1) A *clique* in a graph is a set of vertices every pair of which is joined by an edge. An *independent set* is a set of vertices no pair of which is joined by an edge. Prove that every graph with an infinite set of vertices contains either an infinite clique or an infinite independent set.

This follows from the Infinite Ramsey theorem. If the graph is $G = (V, E)$ then define a colouring $F\ [V]^2 \to \{red, blue\}$ where pairs of points joined by an edge get the colour red, and pairs not joined by an edge get the colour blue. Then an infinite red homogeneous set is a clique, an infinite blue homogeneous set is independent, and by Ramsey one of these two kinds of set must exist.

(2) Consider the following graph. The vertex set is

$$V = \{1/2^n : n \in \mathbb{N}\} \cup \{-1/2^n : n \in \mathbb{N}\}.$$

The edge set is

$$E = \{\{1/2^n, 1/2^{n+1}\} : n \in \mathbb{N}\} \cup \{\{-1/2^n, -1/2^{n+1}\} : n \in \mathbb{N}\}.$$

How many connected components are there?

There are two components. The geometric picture of this graph is irrelevant to graph-theoretic questions about the graph.

(3) (Rather an open-ended one) In class we saw that a triangle-free graph on n vertices has at most $n^2/4$ vertices. Describe a construction that gives for each n a triangle-free graph on n vertices with as many edges as you can manage. You will get

full credit for any construction where the number of edges grows as a quadratic polynomial function of n.

If n is even then we can form a bipartite graph on n vertices by breaking the vertices into two groups A and B each of size $n/2$, and letting $E = \{ab : a \in A, b \in B\}$. This has $n^2/4$ edges. For n odd we can do the same thing with groups of size $(n-1)/2$ and $(n+1)/2$, getting $(n^2-1)/4$ edges. So in either case we can make a bipartite (hence triangle free) graph with at least $(n^2-1)/4$ edges.

(4) Given n vertices, we create a "random graph" by tossing a fair coin $\binom{n}{2}$ times to decide whether each pair of vertices should be joined by an edge. Estimate the probability that the resulting graph is triangle free. What is the expected number of edges?

It is hard to find the exact probability of being triangle free, so we settle for a slightly crude upper bound. Consider a random graph with at least $3n$ vertices. Then we can look at n triples of vertices with no points in common, call them $T_1 \dots T_n$. The probability that T_i ends up being a triangle is $1/8$, so the probability that T_i is not a triangle is $7/8$. It is now easy to see that the probability that none of the T_i are triangles is $(7/8)^n$, which is an upper bound for the probability that our random graph is not triangle free.

Summing up, the probability that a random graph on m vertices is triangle free is at most $(7/8)^{m/3-1}$. This goes to zero quite rapidly, when $m = 200$ we get a bound of about 1.3×10^{-4}.

The expected number of edges is $1/2 \times \binom{n}{2}$.

(5) A graph is a *tree* if it is connected and has no cycles. Find the number of trees on n vertices for $n \leq 5$ (here you should consider two trees to be the same if they are "isomorphic", that is to say there is a bijection between their vertex sets which induces a bijection between their edge sets; to put it more intutively the graphs have the same structure, as an example any two k-cycles are isomorphic). Conjecture a formula for the number of trees on n vertices.

The numbers are $1, 1, 1, 2, 3$. Not really enough for a decent conjecture (though an optimist might say maybe it is $n-2$ for $n \geq 3$). If you compute a few more you get

$$1, 1, 1, 2, 3, 6, 11, 23, 47, 106, 235, 551, 1301, 3159, 7741, 19320$$

You probably started needing a computer somewhere along the way!

There is no easy formula for these numbers. The easiest way of computing them that I know is already pretty complicated and goes like this: Start by computing a sequence of numbers

$$a_{n+1} = (1/n) \times (\sum_{k=1}^{n} (\sum_{d|k} d \times a_d) \times a_{n-k+1}).$$

with $a(1) = 1$.

Let $A(x) = \sum_{n \geq 1} a_n x^n$, let $B(x) = 1 + A(x) - A^2(x)/2 + A(x^2)/2$, and compute the coefficient of x^n in the series expansion of B in powers of x.