DISCRETE MATHEMATICS 21228 — HOMEWORK 5 SOLUTIONS

JC

(1) $R(a,b,c)$ is (by definition) the least N such that every colouring of the pairs from an N-element set in three colours (say red, green and blue) has a homogeneous red set of size a, a homogeneous blue set of size b, or a homogeneous green set of size c.

(a) Find $R(2,2,n)$.
A colouring with no red or blue homogeneous sets of size 2 must colour every pair green. So clearly $R(2,2,n) = n$.

(b) Prove that for $a,b,c > 2$ we have $R(a,b,c) \leq R(a-1,b,c) + R(a,b-1,c) + R(a,b,c-1)$.
We just imitate the induction step in the 2-colour finite Ramsey theorem. Let $N = R(a-1,b,c) + R(a,b-1,c) + R(a,b,c-1)$ and consider a red-blue-green colouring F of the pairs from a set X with N elements. Choose $x \in X$ and partition X as $R \cup B \cup G$, where $R = \{y \in X \setminus \{x\} : F(\{x,y\}) = red\}$ and similarly for B and G with blue and green in place of red.
Clearly at least one of the inequalities $|R| \geq R(a-1,b,c)$, $|B| \geq R(a,b-1,c)$, $|G| \geq R(a,b,c-1)$ must hold. Suppose that $|R| \geq R(a-1,b,c)$. Then there must be $H \subseteq R$ such that H is either red homogeneous of size $a-1$, blue homogeneous of size b or green homogeneous of size c. If H is red homogeneous of size $a-1$ then $H \cup \{x\}$ is red homogeenous of size a, so in all three cases we have a homogeneous set of the required type. The cases $|B| \geq R(a,b-1,c)$ and $|G| \geq R(a,b,c-1)$ are entirely similar.

(2) Find an upper bound for $R(n,n,n)$.
We give a couple of possible bounds. One is better than the other but requires a bit more work.
Bound 1: $R(a,b,c) \leq 3^{a+b+c}$.

Proof: by induction on $a+b+c$, starting from the trivial base case $a=b=c=2$. The successor step works because

$$R(a,b,c) \leq R(a-1,b,c)+R(a,b-1,c)+R(a,b,c-1) \leq 3 \times 3^{a+b+c-1} = 3^{a+b+c}.$$

So $R(n,n,n) \leq 3^{3n}$.

Bound 2: $R(a,b,c) \leq \frac{(a+b+c-3)!}{(a-1)!(b-1)!(c-1)!}$.

Proof: again by induction on $a+b+c$. This gives a bound $R(n,n,n) \leq \frac{(3n-3)!}{((n-1)!)^3}$.

(3) Use the probabilistic method to find a lower bound for $R(n,n,n)$.

We estimate an upper bound for the probability of getting a homogeneous set of size n in a random red-blue-green colouring of pairs from an N-element set. Here there are $3^{\binom{N}{2}}$ possible colourings and we define the probability space so that each colouring is equally likely.

The probability that a given set of size n is homogeneous is $3/3^{\binom{n}{2}}$, so we get an upper bound $\frac{\binom{N}{n}}{3^{\binom{n}{2}-1}}$ for the probability of seeing a homogeneous set of size n.

Accordingly $R(n,n) > N$ for any N such that $\binom{N}{n} < 3^{\binom{n}{2}-1}$, and we need to choose N as large as we can satisfying this inequality. To do this we use the estimates $\binom{N}{n} \leq \frac{N^n}{n!}$ and $n! \geq \sqrt{2\pi n}(n/e)^n$, from which it follows that any N satisfying

$$\frac{N^n}{\sqrt{2\pi n}(n/e)^n} < 3^{\binom{n}{2}-1}$$

will work. A little thought shows that any N with $N \leq \frac{n3^{n/2}}{e\sqrt{3}}$ will satisfy this inequality. We could do a bit better but the main point os to get a lower bound with exponential growth.

(4) Let F be the set of all finite subsets of $\mathbb{N}$. Is the following Ramsey-type statement true or false?

"For all $f : F \longrightarrow \{\text{red}, \text{blue}\}$ there is an infinite set $H \subseteq \mathbb{N}$ such that all the finite subsets of H are given the same colour by f".

False! Just colour even size subsets red and odd size subsets blue.

(5) Prove that if $R(s-1,t)$ and $R(s,t-1)$ are both even then $R(s,t) \leq R(s-1,t) + R(s,t-1) - 1$.

Let X have size $R(s-1,t)+R(s,t-1)-1$, and let F be any red-blue colouring of $[X]^2$. We imitate the proof of the induction step in the finite Ramsey theorem, a bit more carefully since we have one less element to play with.

For each $x \in X$ we may partition $X \ \{x\}$ as $R_x \cup B_x$ where $R_x = \{y \in X \setminus \{x\} : F(\{x, y\}) = red\}$ and $B_x = \{y \in X \setminus \{x\} : F(\{x, y\}) = blue\}$. There are three cases:

Case 1: there is x with $|R_x| \geq R(s-1, t)$. In this case we may proceed as in the proof of the finite Ramsey theorem.

Case 2: there is x with $|B_x| \geq R(s, t-1)$. Again we may proceed as in the proof of the finite Ramsey theorem.

Case 3: for **all** x $|R_x| = R(s-1, t) - 1$ and $|B_x| = R(s, t-1) - 1$. This case seems problematic: we show that it cannot occur.

To see this just observe that each red pair $\{x, y\}$ contributes 1 to each of $|R_x|$ and $|R_y|$; so if A is the number of red pairs then $2A = \sum_x |R_x| = (R(s-1,t)+R(s,t-1)-1) \times (R(s-1,t)-1)$. This is impossible since the RHS is odd while the LHS is even.

(6) (Tricky!) Prove that $R(3, 4) = 9$.

$R(2, 4) = 4$ and $R(3, 3) = 6$ so by the last problem $R(3, 4) \leq 9$. We show $R(3, 4) > 8$ by providing a suitable colouring of 8 points.

We arrange eight points evenly round the circumference of a circle and colour a pair of points red if and only if the points are either adjacent to each other or opposite to each other.

Claim 1: there are no red homogeneous sets of size 3.

Proof: If $\{x, y, z\}$ is such a set then it is not possible that all three points are opposite to each other, so two must be adjacent, say x and y. z must either be adjacent to or opposite to x, and in neither case will it be adjacent or opposite to y.

Claim 2: there are no blue homogeneous sets of size 4.

Proof: such a set can not contain two adjacent points, so it must consist of four of our eight points evenly spaced. But then it contains a pair of opposite points, contradiction!