DISCRETE MATHEMATICS 21228 — HOMEWORK 4

JC

Due in class Wednesday September 24. You may collaborate but *must* write up your solutions by yourself.

Late homework will not be accepted. Homework must either be typed or written legibly in blue or black ink on alternate lines, illegible homework will be returned ungraded (so you can rewrite it legibly).

Please write the name of your recitation instructor and the time and place of your recitation at the top of your homework.

(1) Let f be a random variable on a (finite or countable) probability space X. Assuming that $P(x) > 0$ for all $x \in X$, prove that the variance of f is zero if and only if the random variable f is constant.

(2) Consider randomly colouring a set with k elements in two colours, red and blue, such that points are coloured independently and each point has probability p of getting the colour red and probability $1 - p$ of getting the colour blue. Find the probability that there at least two red points and at least one blue point; for which value of p is this probability largest?

(3) We toss a fair coin repeatedly, and stop when we see two successive throws come up heads. What is the expected value of the number of tosses? Hint: a question from a previous homework and a formula from class may both come in handy.

(4) (The "Drunkard's walk") Consider the following experiment: we generate a random sequence of integers by setting $x_0 = 0$, then $x_{i+1} = x_i + 1$ with probability $1/2$ and $x_{i+1} = x_i - 1$ with probability $1/2$.

Given N, find an expression for the probability that $x_i > 0$ for $0 < i \leq N$. What is the probability that $x_i > 0$ for all $i > 0$?

(5) (This week's Catalan number question) Recall that in class we defined the n^{th} *Catalan number* to be

$$C_n = \frac{1}{n+1}\binom{2n}{n},$$

or equivalently the number of paths from $(0,0)$ to (n,n) which do not go below the line $y = x$.

(a) Show that C_n is the number of ways of dividing a regular $(n+2)$-gon into n triangles by drawing $n-1$ diagonals no two of which meet in the interior of the polygon (they are allowed to meet at vertices).

(b) Find the first six terms of the power series expansion for the function

$$\frac{1-\sqrt{1-4x}}{2x}$$

What do you notice?