DISCRETE MATHEMATICS 21228 — HOMEWORK 3 SOLUTIONS

JC

Due in class Wednesday September 17. You may collaborate but *must* write up your solutions by yourself.

Late homework will not be accepted. Homework must either be typed or written legibly in blue or black ink on alternate lines, illegible homework will be returned ungraded (so you can rewrite it legibly).

Please write the name of your recitation instructor and the time and place of your recitation at the top of your homework.

(1) Consider the experiment in which a biased coin with probability p of coming up heads is tossed N times. Write down a formula for the expectation value of the number of times the coin comes up heads. Prove that this expectation value is Np (why might we have expected this answer?)

As we say in class, the probability of an outcome with a heads is $p^a(1-p)^{N-a}$. There are $\binom{N}{a}$ many such outcomes, and so easily the expectation value of the number of heads is

$$\sum_{a=0}^{N} a \binom{N}{a} p^a (1-p)^{N-a}.$$

There are various tricks we might use to evaluate this expectation. Here is a nice one. Let $q = 1 - p$ and consider the expansion of $(x+q)^N$ by the Binomial theorem, that is

$$(x+q)^N = \sum_{a=0}^{N} \binom{N}{a} x^a q^{N-a}.$$

Differentiate both sides with respect to x and get

$$N(x+q)^{N-1} = \sum_{a=0}^{N} a \binom{N}{a} x^{a-1} q^{N-a}.$$

Multiply by x, set $x = p$, note that $p + q = 1$ and conclude

$$Np = \sum_{a=0}^{N} a \binom{N}{a} p^a q^{N-a}$$

as required.

We might have expected this answer because each time we toss the coin there is a probability p that it comes up heads, so that each of the N tosses makes a contribution p to the total expectation.

Here is another proof which makes this idea precise. For each a let f_a be the function on the probability space of outcomes (strings of N many H's and T's) which is 1 if toss a yields a head and 0 otherwise. Clearly if f is the function which gives the number of heads then $f = f_1 + \dots f_N$, and so by an easy calculation $E(f) = E(f_1) + \dots E(f_N)$.

Now easily $E(f_a)$ is exacty the probability of the event "coin comes up head on toss a", and this probability is p. So $E(f) = Np$.

I gave yet another proof in class, which involved using the identity

$$a\binom{N}{a} = \frac{N!}{(a-1)!(N-a)!} = N\binom{N-1}{a}$$

to rearrange the sum.

(2) Consider the experiment in which the biased coin with probability p of coming up heads is tossed repeatedly until it comes up tails. Find a formula for p_n, the probability that the coin is tossed exactly n times. For a given value of p, what is the least N such that the probability we toss the coin at most N times is at least 0.99?

To be tossed exactly n times, we must have $n-1$ heads then a tail, so $p_n = p^{n-1}(1-p)$. The probability that we toss the coin at most N times is

$$p_1 + \dots p_N = (1-p)(1 + \dots p^{N-1}) = 1 - p^N.$$

Alternative argument: we do not toss the coin at most N times if and only if we come up heads on the first N tries, and this has probability p^N.

Now $1 - p^N \geq 0.99$ if and only if $p^N \leq 0.01$, so easily the least N that works is the least integer such that $N \geq \ln(0.01)/\ln(p)$.

(3) Consider the 3D generalisation of the notion of path: a path is a sequence $(x_0, y_0, z_0), (x_1, y_1, z_1), \dots (x_n, y_n, z_n)$ where at each

step one of the three coordinates increases by one while the other two remain constant. If a, b, c are natural numbers show that the number of paths from $(0, 0, 0)$ to (a, b, c) is

$$\frac{(a+b+c)!}{a!b!c!}$$

Clearly the set of paths can be put in a bijective correspondence with the set of strings of length $a+b+c$ from the alphabet $\{x, y, z\}$ containing a many x's, b many y's and c many z's.

To count these strings, note that if $N = a + b + c$ then there are $\binom{N}{a}$ ways of placing the x's and then $\binom{N-a}{b}$ ways of placing the y's, after which the placing of the z's is determined. This gives

$$\frac{N!}{a!(N-a)!} \times \frac{(N-a)!}{b!(N-a-b)!} = \frac{N!}{a!b!c!}$$

possibilities.

(4) If a, b, c are natural numbers with $c > a + b$, find an expression for the number of paths from $(0, 0, 0)$ to (a, b, c) such that $z > x + y$ for every point (x, y, z) on the path except for $(0, 0, 0)$. (Geometrical picture: we are looking at paths which stay above the plane $z = x + y$).

(Sketchy, you should give a bit more detail) Given a path W from $(0, 0, 0)$ to (a, b, c) such that $z > x + y$ at all but the first point, we define a path W^* from $(0, 0)$ to $(a + b, c)$ as follows: each point (x_i, y_i, z_i) on W corresponds to a point $(x_i + y_i, z_i)$ on W^*. It is easy to see that the path W^* stays above the diagonal.

The key question is now how many values for W correspond to a given value for W^*, that is a given path from $(0, 0)$ to $(a + b, c)$ which stays above the diagonal? Suppose W^* is the path $(s_0, z_0), \ldots (s_N, z_N)$ where $N = a + b + c$. At one of the c steps where $z_{i+1} = z_i + 1$ we must have $x_{i+1} = x_i$ and $y_{i+1} = y_i + 1$, while at one of the $a + b$ steps where $z_{i+1} = z_i$ we must have either $x_{i+1} = x_i + 1$ or $y_{i+1} = y_i + 1$. If we focus on the latter class of steps, we see that we are tracing out a path from $(0, 0)$ to (a, b), and that in fact any such path corresponds to a suitable W.

So we get a total of

$$\frac{c-a-b}{c+a+b}\binom{a+b+c}{c} \times \binom{a+b}{b}.$$

(5) We are given n letters $L_1, \dots L_n$ and n envelopes $E_1, \dots E_n$. How many ways are there of putting the letters in the envelopes so that exactly one letter goes in each envelope? How many ways are there such that for every i, letter L_i does not go into envelope E_i?

There are $n!$ ways of putting the letters in the envelopes. Let d_n be the number of such ways in which every letter goes in the wrong envelope. Permutations of a set X which move every $x \in X$ are called *derangements of* X so we are counting the number of derangements of an n-element set.

Solution one: let π be some derangement of $\{1, \dots n\}$ and suppose that $\pi(1) = i$. If $\pi(i) = 1$ then π gives a derangement of $\{1, \dots n\} \setminus \{1, i\}$, and there are d_{n-2} such derangements.

We claim that the set of derangements with $\pi(1) = i$ and $\pi(i) \neq 1$ is in bijection with the set of derangements of $\{2, \dots n\}$. Given a derangement ρ of $\{2, \dots n\}$ define ρ^* as follows: $\rho^*(1) = i$, $\rho^*(\rho^{-1}(i)) = i$, $\rho^*(x) = \rho(x)$ for all other x.

We conclude that $d_n = (n-1)(d_{n-1} + d_{n-2})$ for all $n > 2$. Clearly $d_1 = 0$ and $d_2 = 1$, so this determines d_n.

Note : to get a closed form expression rather than a recurrence we notice that if $e_n = d_n - nd_{n-1}$ then from the recurrence $e_n = -e_{n-1}$. Also $e_2 = 1$ so by an easy induction $e_n = (-1)^n$. So we get $d_n = nd_{n-1} + (-1)^n$, and then easily $d_n = n! - n!/2! + n!/3! + \dots n!/n!$.

Cultural remark: it is amusing to note that $d_n/n!$ tends to $1/e$ for n large.

Solution 2: use the "inclusion exclusion" formula as in the online notes.

(6) (Challenging) Recall that in class we defined the n^{th} *Catalan number* to be

$$C_n = \frac{1}{n+1}\binom{2n}{n},$$

or equivalently the number of paths from $(0,0)$ to (n,n) which do not go below the line $y = x$.

(a) Show that C_n is the number of sequences $(1, a_1, \dots a_n)$ with $a_i \in \mathbb{N}$ and $a_i \leq i$ and $1 \leq a_1 \leq \dots \leq a_i$.

Argument one (sketch, you should give a bit more detail): consider a path from $(0,0)$ to (n,n) which does not go below the line $y = x$. This path must contain n vertical segments, and the x-coordinate of the n^{th} segment is at most $n-1$ because we are staying above the diagonal. If we let a_i equal the x-coordinate for segment i plus 1 then we can set up a bijection between sequences and paths, so the number of sequences equals the number of paths equals C_n.

(b) Given $2n$ distinct points on the circumference of a circle, C_n is the number of ways of joining them in pairs by drawing n chords of the circle, no two of which intersect. Here is a picture for the case $n = 2$.
Let D_n be the number of ways of drawing the chords and note $D_0 = D_1 = 1$. Label the points with labels from 0 to $2n-1$. It is clear that 0 must be joined to a point with an odd label $2i+1$, because there must be an even number of points on each side as the chords are non-intersecting. It is also clear that there are D_i ways of drawing the chords which connect the $2i$ points $\{1, \ldots 2i\}$ on one side of the chord joining 0 and $2i+1$, and D_{n-i-1} wys of drawing the chords connecting the $2(n-i-1)$ points $\{2i+2, 2n-1\}$ on the other side. So

$$D_n = \sum_{i<n} D_i D_{n-i-1},$$

and since this is the recurrence satisfied by the Catalan numbers we have $D_n = C_n$.